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THE EQUATIONS OF FLUID MECHANICS

EXPRESSED IN CURVILINEAR COORDINATES

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THESIS

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By

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Preface

In this thesis the methods of tensor analysis have been used to obtain expressions for the basic equations of fluid mechanics in terms of several orthogonal curvilinear coordinate systems. While these methods can be used with any valid coordinate transformations, the reader should be cautioned that the specific results obtained here are applicable only to the particular transformations which are listed in Appendix A. There are other methods of defining the various coordinate systems, but, to make use of these differing definitions, the reader would have to start with the basic equations listed in this report and derive his own final results.

It should also be emphasized that this report does not attempt to explain the mechanics of tensor analysis. If the reader is unfamiliar with this branch of mathematics and wishes to gain the background to enable him to fill in the steps which have been omitted in the development of the relationships used in this report, the book by Sokolnikoff (Ref 4) is an excellent text.

The author wishes to express his gratitude to Lt. Ray M. Bowen of the Mechanical Engineering Department of the Institute for the extensive help he has given in the preparation of this thesis. Lt. Bowen is responsible for giving the author his first appreciation of the powers of tensor analysis, and for guiding him through the confusion which occurred from time to time. If there are any errors remaining in this thesis, they are the author's and are not due to any misinformation from Lt. Bowen.

Bradley Sutter

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List of Symbols

ρ	density
v^i	tensor component of the velocity
V_i	physical component of the velocity
g_{ij}	metric tensor, $g_{ij} = 0 , \ i \neq j$
g	determinant of the metric tensor
t	time
u	internal energy per unit mass
h	enthalpy per unit mass
τ^{ij}	stress tensor, $(= -pg^{ij} + \pi^{ij})$
p	pressure
π^{ij}	viscous stress tensor
$\hat{\pi}^{ij}$	physical component of the viscous stress tensor
F^i	tensor component of the body force
$\hat{F}_i$	physical component of the body force
q^i	tensor component of the heat flux vector
$\hat{q}_i$	physical component of the heat flux vector
f^i	tensor component of the acceleration vector
$\hat{f}_i$	physical component of the acceleration vector
η	coefficient of bulk viscosity
σ	coefficient of shear viscosity

Where tensor quantities are defined, the symbols are the same for covariant or contravariant tensors.

Abstract

The applied science of fluid mechanics makes use of three basic equations to analyze and predict the state of a fluid in motion. These equations are the equations of motion, energy, and continuity. In most flow problems, these equations cannot be solved, or are very difficult to solve, unless they are expressed in terms of a coordinate system which conforms to the surface of the duct or body which shapes the flow.

This report utilizes the methods of tensor analysis to transform the basic equations from their Cartesian forms to expressions in ten orthogonal curvilinear coordinate systems. The derivation process is outlined, and the final results are tabulated for each of the coordinate systems. Although this report assumes a Newtonian fluid model, the viscous stress components are listed separately so that, given the proper expressions for the viscous stress components, the results may also be applied to a non-Newtonian fluid.

THE EQUATIONS OF FLUID MECHANICS EXPRESSED IN CURVILINEAR COORDINATES

I. Introduction

Fluid mechanics is defined as the applied science which deals with the principles of both gaseous and liquid flow. Practically, we are concerned with fluid flow over solid bodies or through various types of ducts or channels. In order to predict and describe such flow, we make use of three basic equations: the continuity equation, the energy equation, and the equation of motion. These equations are dependent upon the use of some three-dimensional coordinate system to describe the properties and the movement of a fluid.

For simple flow patterns, such as flow over a flat plate, rectangular Cartesian coordinates are adequate for complete description of the flow characteristics. However, for flow around more complex body shapes, we find that the equations cannot be solved, or are extremely difficult to solve, unless they are expressed in terms of a coordinate system which is compatible with the geometry of the body. In particular, we must have a coordinate system which has a coordinate surface closely approximating the shape of

the body. This is especially important in enabling one to express the boundary conditions in a simple form. Some examples of flow situations which require more sophisticated coordinate systems are: flow over submarine hulls, whose shapes are elongated spheroids; flow through elliptical pipes; and flow through converging-diverging nozzles, in which the nozzle walls approximate hyperboloids of one sheet.

The problem is one of expressing the familiar Cartesian forms of the basic equations in terms of the various curvilinear coordinate systems. To do this, the methods of tensor analysis will be used. The general approach has been outlined for the continuity and motion equations by McConnell (Ref 1: 271-313), among others, and results have been obtained for cylindrical and spherical coordinates. The methods used in this report differ only slightly from those used by McConnell.

The fluid model is assumed to be viscous, heat conducting, and isotropic. Chemical, electromagnetic, radiation, and diffusion effects are ignored. The fluid is also assumed to be Newtonian. However, the expressions for the viscous stress tensor are listed separately so that, given the proper expressions for the viscous stress tensor, the results may also be applied to non-Newtonian fluids.

II. The Basic Equations

The basic equations are presented here in fairly conventional form, and they are then adapted to a form more suitable for final tabulation. If the reader is interested in the derivation of the equations, he is referred to Appendix B.

Continuity Equation

The continuity equation can be expressed in tensor notation as

$$\frac{\partial \rho}{\partial t} + (\rho v^i)_{,i} = 0 \quad , \tag{1}$$

where v^i is the i'th component of the velocity vector, and ρ is the density. The second term represents the divergence of a vector, and it can be expressed more conveniently by using

$$(\rho v^i)_{,i} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\frac{\sqrt{g}}{\sqrt{g_{(i)(i)}}}\, \rho V_i\right) \quad , \tag{2}$$

where V_i represents the physical component of the velocity vector. Substituting equation (2) into equation (1), we obtain the continuity equation in its final form:

$$\frac{\partial \rho}{\partial t} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\frac{\sqrt{g}}{\sqrt{g_{(i)(i)}}}\, \rho V_i\right) = 0 \quad . \tag{3}$$

Energy Equation

In tensor notation, one form of the energy equation is

$$\rho\left[\frac{\partial}{\partial t}\left(h + \frac{1}{2} v_i v^i\right) + v^j\left(h + \frac{1}{2} v_i v^i\right)_{,j}\right] = \frac{\partial p}{\partial t} +$$

$$(v_i \pi^{ij})_{,j} + \rho v_i F^i - g^i_{,i} \quad , \qquad (4)$$

where h is enthalpy, π^{ij} is the ij'th component of the viscous stress tensor, p is pressure, and g^i is the i'th component of the heat flux vector. By substituting physical components and making use of the general form of equation (2), we obtain the energy equation in its final form:

$$\rho\left[\frac{\partial}{\partial t}\left(h + \frac{1}{2} V_i V_i\right) + \frac{V_j}{\sqrt{g_{(j)(j)}}} \frac{\partial}{\partial x^j}\left(h + \frac{1}{2} V_i V_i\right)\right] = \frac{\partial p}{\partial t} +$$

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j}\left(\frac{\sqrt{g}}{\sqrt{g_{(j)(j)}}} V_i \hat{\pi}^{ij}\right) + \rho V_i \hat{F}_i - \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\frac{\sqrt{g}}{\sqrt{g_{(i)(i)}}} \hat{g}_i\right) \quad , \qquad (5)$$

where $\hat{\pi}^{ij}$, $\hat{F}_i$, and $\hat{g}_i$ are the physical components of the viscous stress tensor, the body force, and the heat flux vector, respectively. The ij'th component of the viscous stress tensor can be written, after Sokolnikoff (Ref 4: 321-324), as

$$\pi^{ij} \equiv (\eta - \tfrac{2}{3}\sigma)\, g^{ij} v^{m}_{\ ,m} +$$

$$\sigma \left(g^{im} v^{j}_{\ ,m} + g^{jm} v^{i}_{\ ,m} \right) \quad , \tag{6}$$

where η is the coefficient of bulk viscosity (usually taken to be zero), and σ is the coefficient of shear viscosity. After carrying out the indicated operations and substituting the physical components for the velocity vectors, we obtain

$$\pi^{ij} = (\eta - \tfrac{2}{3}\sigma) \frac{g^{ij}}{\sqrt{g}} \frac{\partial}{\partial x^m} \left(\frac{\sqrt{g}}{\sqrt{g_{(m)(m)}}} V_m \right) +$$

$$\sigma \left\{ g^{im} \frac{\partial}{\partial x^m} \left(\frac{V_j}{\sqrt{g_{(j)(j)}}} \right) + g^{jm} \frac{\partial}{\partial x^m} \left(\frac{V_i}{\sqrt{g_{(i)(i)}}} \right) + \right.$$

$$\left. \frac{1}{2} \frac{\partial g_{m\beta}}{\partial x^\alpha} \left(g^{im} g^{j\beta} + g^{jm} g^{i\beta} \right) \frac{V_\alpha}{\sqrt{g_{(\alpha)(\alpha)}}} \right\} \quad . \tag{7}$$

We can obtain the physical components of π^{ij} by multiplying equation (7) by $\sqrt{g_{(i)(i)} g_{(j)(j)}}$. Although the multiplication will not be shown here, the physical components, rather than the tensor components, will be tabulated in the next section.

Equation of Motion

The tensor form of the equation of motion is

$$\rho\left(f_i - F_i\right) = g^{jk}\tau_{ij,k} \quad , \qquad (8)$$

where f_i is the i'th component of the acceleration vector, and τ_{ij} represents the viscous and pressure forces as follows:

$$\tau_{ij} = -p g_{ij} + \pi_{ij} \quad . \qquad (9)$$

The physical component of the acceleration vector is

$$\hat{f}_i = \sqrt{g^{(i)(i)}}\, f_{(i)} = \frac{1}{\sqrt{g_{(i)(i)}}}\, f_{(i)} \quad , \qquad (10)$$

and the physical component of the body force is

$$\hat{F}_i = \sqrt{g^{(i)(i)}}\, F_{(i)} = \frac{1}{\sqrt{g_{(i)(i)}}}\, F_{(i)} \quad , \qquad (11)$$

where $\hat{f}_i$ and $\hat{F}_i$ represent the physical component of the acceleration vector and the body force, respectively. By substituting equations (10) and (11) into equation (9), we obtain

$$\rho\left(\hat{f}_i - \hat{F}_i\right) = \frac{g^{jk}}{\sqrt{g_{(i)(i)}}}\, \tau_{(i)j,k} \qquad (12)$$

which is the final form of the equation of motion.

The i'th component of the acceleration vector is

$$\hat{f}_i = \frac{1}{\sqrt{g_{(i)(i)}}}\left[\frac{\partial v_{(i)}}{\partial t} + v_{(i),j}\, v^{j}\right] \quad . \tag{13}$$

The final form of the acceleration vector is obtained by expanding equation (13) and substituting the physical components of the velocity vectors:

$$\hat{f}_i = \frac{\partial V_i}{\partial t} + \frac{1}{\sqrt{g_{(i)(i)}}}\left[V_j \frac{\partial V_j}{\partial x^{(i)}} + \right.$$

$$\left.\frac{V_j}{\sqrt{g_{(j)(j)}}}\left\{\frac{\partial}{\partial x^{j}}\left(\sqrt{g_{(i)(i)}}\, V_{(i)}\right) - \frac{\partial}{\partial x^{i}}\left(\sqrt{g_{(j)(j)}}\, V_j\right)\right\}\right] \quad . \tag{14}$$

The term on the right side of equation (12) can be expanded, using equation (9), as follows:

$$\frac{1}{\sqrt{g_{(i)(i)}}}\, g^{jk}\, T_{(i)j,k} = \frac{1}{\sqrt{g_{(i)(i)}}}\left(-p_{,(i)} + g_{(i)j}\, \pi^{jk}{}_{,k}\right) \quad , \tag{15}$$

where

$$g_{ij}\, \pi^{jk}{}_{,k} = g_{ij}\left[\frac{\partial}{\partial x^{k}}\left(\pi^{jk}\right) + \right.$$

$$\left.\left\{ {j \atop \alpha\, k} \right\} \pi^{\alpha k} + \left\{ {k \atop \alpha\, k} \right\} \pi^{j\alpha}\right] \quad . \tag{16}$$

Expanding equation (16) to eliminate the Christoffel sym-

bols, we obtain, after simplifying

$$g_{ij}\, \pi^{jk}{}_{,k} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k}\left(\sqrt{g}\, g_{ij}\, \pi^{jk}\right) - \frac{1}{2} \frac{\partial}{\partial x^i}\left(g_{jk}\right) \pi^{jk} \quad . \tag{17}$$

Substituting equation (17) into equation (15), we obtain

$$\frac{g^{jk}}{\sqrt{g_{(i)(i)}}}\, \tau_{(i)j,k} = \frac{1}{\sqrt{g_{(i)(i)}}} \left[-p_{,(i)} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k}\left(\sqrt{g}\, g_{(i)j}\, \pi^{jk}\right) - \frac{1}{2} \frac{\partial g_{jk}}{\partial x^{(i)}}\, \pi^{jk} \right] . \tag{18}$$

Substitution of the physical components of the viscous stress tensor yields

$$\frac{g^{jk}}{\sqrt{g_{(i)(i)}}}\, \tau_{(i)j,k} = \frac{1}{\sqrt{g_{(i)(i)}}} \left[-p_{,(i)} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k}\left(\frac{\sqrt{g}\, g_{(i)j}}{\sqrt{g_{(j)(j)}\, g_{(k)(k)}}}\, \hat{\pi}^{jk}\right) - \frac{1}{2\sqrt{g_{(j)(j)}\, g_{(k)(k)}}} \frac{\partial g_{jk}}{\partial x^{(i)}}\, \hat{\pi}^{jk} \right] . \tag{19}$$

III. The Completed Transformations

For convenience, the equations that are to be expressed in the various coordinate systems have been restated on a fold-out sheet in Appendix C.

Cartesian Coordinates

$$x^1 = x \qquad x^2 = y \qquad x^3 = z \tag{20}$$

$$g_{11} = g_{22} = g_{33} = g = 1 \tag{21}$$

Continuity Equation.

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho V_x) + \frac{\partial}{\partial y}(\rho V_y) + \frac{\partial}{\partial z}(\rho V_z) = 0 \tag{22}$$

Energy Equation. The left side of equation (5) becomes

$$\rho \left\{ \frac{\partial}{\partial t}\left[h + \frac{1}{2}(V_x^2 + V_y^2 + V_z^2)\right] + V_x \frac{\partial}{\partial x}\left[h + \frac{1}{2}(V_x^2 + V_y^2 + V_z^2)\right] + V_y \frac{\partial}{\partial y}\left[h + \frac{1}{2}(V_x^2 + V_y^2 + V_z^2)\right] + V_z \frac{\partial}{\partial z}\left[h + \frac{1}{2}(V_x^2 + V_y^2 + V_z^2)\right] \right\} . \tag{23}$$

The remaining terms of interest are

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\frac{\sqrt{g}}{\sqrt{g_{(j)(j)}}}V_i\hat{\pi}^{ij}\right) = \frac{\partial}{\partial x}\left(V_x\hat{\pi}^{xx} + V_y\hat{\pi}^{yx} + V_z\hat{\pi}^{zx}\right) + \frac{\partial}{\partial y}\left(V_x\hat{\pi}^{xy} + V_y\hat{\pi}^{yy} + V_z\hat{\pi}^{zy}\right) + \frac{\partial}{\partial z}\left(V_x\hat{\pi}^{xz} + V_y\hat{\pi}^{yz} + V_z\hat{\pi}^{zz}\right) , \tag{24}$$

$$\rho V_i\hat{F}_i = \rho\left(V_x\hat{F}_x + V_y\hat{F}_y + V_z\hat{F}_z\right) , \tag{25}$$

and

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}\,\frac{\hat{q}_i}{\sqrt{g_{(i)(i)}}}\right) = \frac{\partial}{\partial x}(\hat{q}_x) + \frac{\partial}{\partial y}(\hat{q}_y) + \frac{\partial}{\partial z}(\hat{q}_z) . \tag{26}$$

The components of the viscous stress tensor are

$$\hat{\pi}^{xy} = \hat{\pi}^{yx} = \sigma\left(\frac{\partial V_y}{\partial x} + \frac{\partial V_x}{\partial y}\right) , \tag{27}$$

$$\hat{\pi}^{xz} = \hat{\pi}^{zx} = \sigma\left(\frac{\partial V_z}{\partial x} + \frac{\partial V_x}{\partial z}\right) , \tag{28}$$

$$\hat{\pi}^{yz} = \hat{\pi}^{zy} = \sigma\left(\frac{\partial V_z}{\partial y} + \frac{\partial V_y}{\partial z}\right) , \tag{29}$$

$$\hat{\pi}^{xx} = \left(\eta - \tfrac{2}{3}\sigma\right)\left[\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right] + 2\sigma \frac{\partial V_x}{\partial x} \quad , \tag{30}$$

$$\hat{\pi}^{yy} = \left(\eta - \tfrac{2}{3}\sigma\right)\left[\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right] + 2\sigma \frac{\partial V_y}{\partial y} \quad , \tag{31}$$

and

$$\hat{\pi}^{zz} = \left(\eta - \tfrac{2}{3}\sigma\right)\left[\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right] + 2\sigma \frac{\partial V_z}{\partial z} \quad . \tag{32}$$

Equation of Motion. The components of the viscous stress tensor are tabulated above. The remaining terms of interest are

$$\hat{f}_x = \frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} + V_z \frac{\partial V_x}{\partial z} \quad , \tag{33}$$

$$\hat{f}_y = \frac{\partial V_y}{\partial t} + V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y} + V_z \frac{\partial V_y}{\partial z} \quad , \tag{34}$$

$$\hat{f}_z = \frac{\partial V_z}{\partial t} + V_x \frac{\partial V_z}{\partial x} + V_y \frac{\partial V_z}{\partial y} + V_z \frac{\partial V_z}{\partial z} \quad , \tag{35}$$

$$\frac{g^{jk}}{\sqrt{g_{11}}} \tau_{1j,k} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}(\hat{\pi}^{xx}) +$$

$$\frac{\partial}{\partial y}(\hat{\pi}^{xy}) + \frac{\partial}{\partial z}(\hat{\pi}^{xz}) , \qquad (36)$$

$$\frac{g^{jk}}{\sqrt{g_{22}}} \tau_{2j,k} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}(\hat{\pi}^{yx}) +$$

$$\frac{\partial}{\partial y}(\hat{\pi}^{yy}) + \frac{\partial}{\partial z}(\hat{\pi}^{yz}) , \qquad (37)$$

and

$$\frac{g^{jk}}{\sqrt{g_{33}}} \tau_{3j,k} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}(\hat{\pi}^{zx}) +$$

$$\frac{\partial}{\partial y}(\hat{\pi}^{zy}) + \frac{\partial}{\partial z}(\hat{\pi}^{zz}) . \qquad (38)$$

Cylindrical Coordinates

$$X^1 = r \qquad X^2 = \theta \qquad X^3 = Z \qquad (39)$$

$$g_{11} = g_{33} = 1 \qquad (40)$$

$$g_{22} = g = r^2 \qquad (41)$$

Continuity Equation.

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho V_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho V_\theta) + \frac{\partial}{\partial z}(\rho V_z) = 0 \tag{42}$$

Energy Equation. The left side of equation (5) becomes

$$\rho\Big\{\frac{\partial}{\partial t}\Big[h + \frac{1}{2}(V_r^2 + V_\theta^2 + V_z^2)\Big] + V_r\frac{\partial}{\partial r}\Big[h + \frac{1}{2}(V_r^2 + V_\theta^2 + V_z^2)\Big] + \frac{1}{r}V_\theta\frac{\partial}{\partial \theta}\Big[h + \frac{1}{2}(V_r^2 + V_\theta^2 + V_z^2)\Big] + V_z\frac{\partial}{\partial z}\Big[h + \frac{1}{2}(V_r^2 + V_\theta^2 + V_z^2)\Big]\Big\} . \tag{43}$$

Also,

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(\frac{\sqrt{g}}{\sqrt{g_{(j)(j)}}}V_i\pi^{ij}\right) = \frac{1}{r}\frac{\partial}{\partial r}\Big[r(V_r\hat{\pi}^{rr} + V_\theta\hat{\pi}^{\theta r} + V_z\hat{\pi}^{zr})\Big] + \frac{1}{r}\frac{\partial}{\partial \theta}\Big[V_r\hat{\pi}^{r\theta} + V_\theta\hat{\pi}^{\theta\theta} + V_z\hat{\pi}^{z\theta}\Big] + \frac{\partial}{\partial z}\left(V_r\hat{\pi}^{rz} + V_\theta\hat{\pi}^{\theta z} + V_z\hat{\pi}^{zz}\right) , \tag{44}$$

$$\rho V_i\hat{F}_i = \rho\left(V_r\hat{F}_r + V_\theta\hat{F}_\theta + V_z\hat{F}_z\right) , \tag{45}$$

and

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\frac{\sqrt{g}}{\sqrt{g_{(i)(i)}}}\,\hat{q}_i\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\hat{q}_r\right) +$$

$$\frac{1}{r}\frac{\partial}{\partial \theta}\left(\hat{q}_\theta\right) + \frac{\partial}{\partial z}\left(\hat{q}_z\right) \quad . \tag{46}$$

The components of the stress tensor are

$$\hat{\pi}^{r\theta} = \hat{\pi}^{\theta r} = \tau\left[r\frac{\partial}{\partial r}\left(\frac{1}{r}V_\theta\right) + \frac{1}{r}\frac{\partial}{\partial \theta}(V_r)\right] \quad , \tag{47}$$

$$\hat{\pi}^{rz} = \hat{\pi}^{zr} = \tau\left[\frac{\partial V_z}{\partial r} + \frac{\partial V_r}{\partial z}\right] \quad , \tag{48}$$

$$\hat{\pi}^{\theta z} = \hat{\pi}^{z\theta} = \tau\left[\frac{1}{r}\frac{\partial V_z}{\partial \theta} + \frac{\partial V_\theta}{\partial z}\right] \quad , \tag{49}$$

$$\hat{\pi}^{rr} = (\eta - \tfrac{2}{3}\tau)\left[\frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} +\right.$$

$$\left.\frac{\partial V_z}{\partial z}\right] + 2\tau\frac{\partial V_r}{\partial r} \quad , \tag{50}$$

$$\hat{\pi}^{\theta\theta} = (\eta - \tfrac{2}{3}\tau)\left[\frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} +\right.$$

$$\left.\frac{\partial V_z}{\partial z}\right] + 2\tau\left[\frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{1}{r}V_r\right] \quad , \tag{51}$$

and

$$\hat{\pi}^{zz} = (\eta - \tfrac{2}{3}\tau)\left[\frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z}\right] + 2\tau \frac{\partial V_z}{\partial z} . \tag{52}$$

<u>Equation</u> <u>of</u> <u>Motion</u>. In addition to the components of the stress tensor tabulated above, the terms of interest are

$$\hat{f}_r = \frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + \frac{1}{r} V_\theta \frac{\partial V_r}{\partial \theta} + V_z \frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r} , \tag{53}$$

$$\hat{f}_\theta = \frac{\partial V_\theta}{\partial t} + \frac{V_r}{r}\frac{\partial}{\partial r}(rV_\theta) + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + V_z \frac{\partial V_\theta}{\partial z} , \tag{54}$$

$$\hat{f}_z = \frac{\partial V_z}{\partial t} + V_r \frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z \frac{\partial V_z}{\partial z} , \tag{55}$$

$$\frac{g^{jk}}{\sqrt{g_{11}}} \tau_{1j,k} = -\frac{\partial p}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r}(r\hat{\pi}^{rr}) + \frac{1}{r}\frac{\partial}{\partial \theta}(\hat{\pi}^{r\theta}) + \frac{\partial}{\partial z}(\hat{\pi}^{rz}) - \frac{1}{r}\hat{\pi}^{\theta\theta} , \tag{56}$$

$$\frac{g^{jk}}{\sqrt{g_{22}}} \tau_{2j,k} = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{\partial}{\partial r}(\hat{\pi}^{\theta r}) + \frac{1}{r}\frac{\partial}{\partial \theta}(\hat{\pi}^{\theta\theta}) + \frac{\partial}{\partial z}(\hat{\pi}^{\theta z}) + \frac{2}{r}\hat{\pi}^{\theta r} , \tag{57}$$

and

$$\frac{g^{jk}}{\sqrt{g_{33}}} \hat{\Upsilon}_{3j,k} = -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\hat{\pi}^{zr}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\hat{\pi}^{z\theta}\right) +$$

$$\frac{\partial}{\partial z}\left(\hat{\pi}^{zz}\right) \quad . \tag{58}$$

Spherical Coordinates

$$x^1 = r \qquad x^2 = \theta \qquad x^3 = \varphi \tag{59}$$

$$g_{11} = 1 \tag{60}$$

$$g_{22} = r^2 \tag{61}$$

$$g_{33} = r^2 \sin^2\theta \tag{62}$$

$$g = r^4 \sin^2\theta \tag{63}$$

Continuity Equation.

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho V_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\, \rho V_\theta\right) +$$

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\left(\rho V_\varphi\right) = 0 \tag{64}$$

Energy Equation. The left side of equation (5) becomes

$$\rho\left\{\frac{\partial}{\partial t}\left[h+\frac{1}{2}(V_r^2+V_\theta^2+V_\varphi^2)\right]+V_r\frac{\partial}{\partial r}\left[h+\frac{1}{2}(V_r^2+V_\theta^2+V_\varphi^2)\right]+\frac{V_\theta}{r}\frac{\partial}{\partial\theta}\left[h+\frac{1}{2}(V_r^2+V_\theta^2+V_\varphi^2)\right]+\frac{V_\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\left[h+\frac{1}{2}(V_r^2+V_\theta^2+V_\varphi^2)\right]\right\}. \qquad (65)$$

Also,

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(\frac{\sqrt{g}}{\sqrt{g_{(j)(j)}}}V_i\hat{\pi}^{ij}\right)=\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2(V_r\hat{\pi}^{rr}+V_\theta\hat{\pi}^{\theta r}+V_\varphi\hat{\pi}^{\varphi r})\right]+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\,(V_r\hat{\pi}^{r\theta}+V_\theta\hat{\pi}^{\theta\theta}+V_\varphi\hat{\pi}^{\varphi\theta})\right]+\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left[V_r\hat{\pi}^{r\varphi}+V_\theta\hat{\pi}^{\theta\varphi}+V_\varphi\hat{\pi}^{\varphi\varphi}\right], \qquad (66)$$

$$\rho V_i\hat{F}_i=\rho\left[V_r\hat{F}_r+V_\theta\hat{F}_\theta+V_\varphi\hat{F}_\varphi\right], \qquad (67)$$

and

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\frac{\sqrt{g}}{\sqrt{g_{(i)(i)}}}\hat{q}_i\right)=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\hat{q}_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,\hat{q}_\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}(\hat{q}_\varphi). \qquad (68)$$

The components of the viscous stress tensor are

$$\hat{\Pi}^{r\theta} = \hat{\Pi}^{\theta r} = \tau\left[r\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right) + \frac{1}{r}\frac{\partial V_r}{\partial \theta}\right], \tag{69}$$

$$\hat{\Pi}^{r\varphi} = \hat{\Pi}^{\varphi r} = \tau\left[r\frac{\partial}{\partial r}\left(\frac{V_\varphi}{r}\right) + \frac{1}{r\sin\theta}\frac{\partial V_r}{\partial \varphi}\right], \tag{70}$$

$$\hat{\Pi}^{\theta\varphi} = \hat{\Pi}^{\varphi\theta} = \tau\left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{V_\varphi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial V_\theta}{\partial\varphi}\right], \tag{71}$$

$$\hat{\Pi}^{rr} = \left(\eta - \tfrac{2}{3}\tau\right)\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(V_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\varphi}{\partial\varphi}\right] + 2\tau\frac{\partial V_r}{\partial r}, \tag{72}$$

$$\hat{\Pi}^{\theta\theta} = \left(\eta - \tfrac{2}{3}\tau\right)\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(V_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\varphi}{\partial\varphi}\right. + 2\tau\left[\frac{1}{r}\left(\frac{\partial V_\theta}{\partial\theta} + V_r\right)\right], \tag{73}$$

and

$$\hat{\Pi}^{\varphi\varphi} = \left(\eta - \tfrac{2}{3}\tau\right)\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(V_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\varphi}{\partial\varphi}\right] + 2\tau\left[\frac{1}{r\sin\theta}\left(\frac{\partial V_\varphi}{\partial\varphi} + V_r + V_\theta\cos\theta\right)\right], \tag{74}$$

<u>Equation of Motion</u>. The terms of interest, in addition to the components of the viscous stress tensor already listed, are

$$\hat{f}_r = \frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + \frac{V_\varphi}{r \sin\theta}\frac{\partial V_r}{\partial \varphi} - \frac{V_\theta^2 + V_\varphi^2}{r} , \tag{75}$$

$$\hat{f}_\theta = \frac{\partial V_\theta}{\partial t} + \frac{V_r}{r}\frac{\partial}{\partial r}(r V_\theta) + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_\varphi}{r \sin\theta}\frac{\partial V_\theta}{\partial \varphi} - \frac{V_\varphi^2}{r}\cot\theta , \tag{76}$$

$$\hat{f}_\varphi = \frac{\partial V_\varphi}{\partial t} + \frac{V_r}{r}\frac{\partial}{\partial r}(r V_\varphi) + \frac{V_\theta}{r \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, V_\varphi) + \frac{V_\varphi}{r \sin\theta}\frac{\partial V_\varphi}{\partial \varphi} , \tag{77}$$

$$\frac{g^{jk}}{\sqrt{g_{11}}}\tau_{1j,k} = -\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \hat{\pi}^{rr}) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, \hat{\pi}^{r\theta}) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \varphi}(\hat{\pi}^{r\varphi}) - \frac{1}{r}(\hat{\pi}^{\theta\theta} + \hat{\pi}^{\varphi\varphi}) , \tag{78}$$

$$\frac{g^{jk}}{\sqrt{g_{22}}} T_{2j,k} = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\hat{\pi}^{\theta r}\right) +$$

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\hat{\pi}^{\theta\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(\hat{\pi}^{\theta\varphi}\right) -$$

$$\hat{\pi}^{\varphi\varphi}\cot\theta + 2\hat{\pi}^{\theta r}\,, \qquad (79)$$

and

$$\frac{g^{jk}}{\sqrt{g_{33}}} T_{3j,k} = -\frac{1}{r\sin\theta}\frac{\partial p}{\partial\varphi} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\hat{\pi}^{\varphi r}\right) +$$

$$\frac{1}{r}\frac{\partial}{\partial\theta}\left(\hat{\pi}^{\varphi\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(\hat{\pi}^{\varphi\varphi}\right) +$$

$$\frac{2}{r}\hat{\pi}^{\varphi r} + \frac{2\cot\theta}{r}\hat{\pi}^{\varphi\theta}\,. \qquad (80)$$

<u>Parabolic Coordinates</u>

$$X^1 = \lambda \qquad X^2 = \mu \qquad X^3 = \varphi \qquad (81)$$

$$g_{11} = \frac{\lambda+\mu}{4\lambda} \qquad (82)$$

$$g_{22} = \frac{\lambda+\mu}{4\mu} \qquad (83)$$

$$g_{33} = \lambda\mu \qquad (84)$$

$$g = \frac{(\lambda+\mu)^2}{16} \tag{85}$$

Continuity Equation.

$$\frac{\partial \rho}{\partial t} + \frac{2}{\lambda+\mu}\frac{\partial}{\partial \lambda}\left[\sqrt{\lambda(\lambda+\mu)}\;\rho V_\lambda\right] +$$
$$\frac{2}{\lambda+\mu}\frac{\partial}{\partial \mu}\left[\sqrt{\mu(\lambda+\mu)}\;\rho V_\mu\right] +$$
$$\frac{1}{\sqrt{\lambda\mu}}\frac{\partial}{\partial \varphi}\left[\rho V_\varphi\right] = 0 \tag{86}$$

Energy Equation. The left side of equation (5) becomes

$$\rho\left\{\frac{\partial}{\partial t}\left[h + \frac{1}{2}(V_\lambda^2 + V_\mu^2 + V_\varphi^2)\right] + 2V_\lambda\sqrt{\frac{\lambda}{\lambda+\mu}}\frac{\partial}{\partial \lambda}\left[h + \right.\right.$$
$$\left.\frac{1}{2}(V_\lambda^2 + V_\mu^2 + V_\varphi^2)\right] + 2V_\mu\sqrt{\frac{\mu}{\lambda+\mu}}\frac{\partial}{\partial \mu}\left[h + \frac{1}{2}(V_\lambda^2 + V_\mu^2 + \right.$$
$$\left.\left.V_\varphi^2)\right] + \frac{V_\varphi}{\sqrt{\lambda\mu}}\frac{\partial}{\partial \varphi}\left[h + \frac{1}{2}(V_\lambda^2 + V_\mu^2 + V_\varphi^2)\right]\right\} \quad . \tag{87}$$

Also,

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j}\left(\frac{\sqrt{g}}{\sqrt{g_{(j)(j)}}} V_i \hat{\pi}^{ij}\right) = \frac{2}{\lambda+\mu} \frac{\partial}{\partial \lambda}\Big[\sqrt{\lambda(\lambda+\mu)}\,\Big(V_\lambda \hat{\pi}^{\lambda\lambda} +$$

$$V_\mu \hat{\pi}^{\mu\lambda} + V_\varphi \hat{\pi}^{\varphi\lambda}\Big)\Big] + \frac{2}{\lambda+\mu} \frac{\partial}{\partial \mu}\Big[\sqrt{\mu(\lambda+\mu)}\,\Big(V_\lambda \hat{\pi}^{\lambda\mu} +$$

$$V_\mu \hat{\pi}^{\mu\mu} + V_\varphi \hat{\pi}^{\varphi\mu}\Big)\Big] + \frac{1}{\sqrt{\lambda\mu}} \frac{\partial}{\partial \varphi}\Big[V_\lambda \hat{\pi}^{\lambda\varphi} + V_\mu \hat{\pi}^{\mu\varphi} + V_\varphi \hat{\pi}^{\varphi\varphi}\Big] \quad , \qquad (88)$$

$$\rho V_i \hat{F}_i = \rho\left[V_\lambda \hat{F}_\lambda + V_\mu \hat{F}_\mu + V_\varphi \hat{F}_\varphi\right] \quad , \qquad (89)$$

and

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\frac{\sqrt{g}}{\sqrt{g_{(i)(i)}}}\, \hat{q}_i\right) = \frac{2}{\lambda+\mu} \frac{\partial}{\partial \lambda}\left[\sqrt{\lambda(\lambda+\mu)}\;\hat{q}_\lambda\right] +$$

$$\frac{2}{\lambda+\mu} \frac{\partial}{\partial \mu}\left[\sqrt{\mu(\lambda+\mu)}\;\hat{q}_\mu\right] + \frac{1}{\sqrt{\lambda\mu}} \frac{\partial}{\partial \varphi}\left(\hat{q}_\varphi\right) \quad . \qquad (90)$$

The components of the viscous stress tensor are

$$\hat{\pi}^{\lambda\mu} = \hat{\pi}^{\mu\lambda} = \sigma\left[2\sqrt{\lambda}\, \frac{\partial}{\partial \lambda}\left(\frac{V_\mu}{\sqrt{\lambda+\mu}}\right) +\right.$$

$$\left. 2\sqrt{\mu}\, \frac{\partial}{\partial \mu}\left(\frac{V_\lambda}{\sqrt{\lambda+\mu}}\right)\right] \quad , \qquad (91)$$

$$\hat{\pi}^{\lambda\varphi} = \hat{\pi}^{\varphi\lambda} = \sigma\left[\frac{2\lambda}{\sqrt{\lambda+\mu}}\, \frac{\partial}{\partial \lambda}\left(\frac{V_\varphi}{\sqrt{\lambda}}\right) +\right.$$

$$\left. \frac{1}{\sqrt{\lambda\mu}}\, \frac{\partial V_\lambda}{\partial \varphi}\right] \quad , \qquad (92)$$

$$\hat{\Pi}^{\mu\varphi} = \hat{\Pi}^{\varphi\mu} = \tau\left[\frac{2\mu}{\sqrt{\lambda+\mu}}\frac{\partial}{\partial\mu}\left(\frac{V_\varphi}{\sqrt{\mu}}\right) + \frac{1}{\sqrt{\lambda\mu}}\frac{\partial V_\mu}{\partial\varphi}\right], \tag{93}$$

$$\hat{\Pi}^{\lambda\lambda} = (\eta - \tfrac{2}{3}\tau)\left[\frac{2}{\lambda+\mu}\frac{\partial}{\partial\lambda}\left(\sqrt{\lambda(\lambda+\mu)}\,V_\lambda\right) + \frac{2}{\lambda+\mu}\frac{\partial}{\partial\mu}\left(\sqrt{\mu(\lambda+\mu)}\,V_\mu\right) + \frac{1}{\sqrt{\lambda\mu}}\frac{\partial V_\varphi}{\partial\varphi}\right] + \tau\left[4\frac{\partial}{\partial\lambda}\left(\sqrt{\frac{\lambda}{\lambda+\mu}}\,V_\lambda\right) + \frac{2\sqrt{\mu}}{(\lambda+\mu)^{3/2}}V_\mu - \frac{2\mu}{\sqrt{\lambda}\,(\lambda+\mu)^{3/2}}V_\lambda\right], \tag{94}$$

$$\hat{\Pi}^{\mu\mu} = (\eta - \tfrac{2}{3}\tau)\left[\frac{2}{\lambda+\mu}\frac{\partial}{\partial\lambda}\left(\sqrt{\lambda(\lambda+\mu)}\,V_\lambda\right) + \frac{2}{\lambda+\mu}\frac{\partial}{\partial\mu}\left(\sqrt{\mu(\lambda+\mu)}\,V_\mu\right) + \frac{1}{\sqrt{\lambda\mu}}\frac{\partial V_\varphi}{\partial\varphi}\right] + \tau\left[4\frac{\partial}{\partial\mu}\left(\sqrt{\frac{\mu}{\lambda+\mu}}\,V_\mu\right) + \frac{2\sqrt{\lambda}}{(\lambda+\mu)^{3/2}}V_\lambda - \frac{2\lambda}{\sqrt{\mu}\,(\lambda+\mu)^{3/2}}V_\mu\right], \tag{95}$$

and

$$\hat{\Pi}^{\varphi\varphi} = (\eta - \tfrac{2}{3}\tau)\left[\frac{2}{\lambda+\mu}\frac{\partial}{\partial\lambda}\left(\sqrt{\lambda(\lambda+\mu)}\,V_\lambda\right) + \frac{2}{\lambda+\mu}\frac{\partial}{\partial\mu}\left(\sqrt{\mu(\lambda+\mu)}\,V_\mu\right) + \frac{1}{\sqrt{\lambda\mu}}\frac{\partial V_\varphi}{\partial\varphi}\right] + \tau\left[\frac{2}{\sqrt{\lambda\mu}}\frac{\partial V_\varphi}{\partial\varphi} + \frac{2}{\sqrt{\lambda(\lambda+\mu)}}V_\lambda + \frac{2}{\sqrt{\mu(\lambda+\mu)}}V_\mu\right]. \tag{96}$$

Equations of Motion. The components of the viscous stress tensor are listed above. The remaining terms of interest are

$$\hat{f}_\lambda = \frac{\partial V_\lambda}{\partial t} + 2V_\lambda\sqrt{\frac{\lambda}{\lambda+\mu}}\,\frac{\partial V_\lambda}{\partial \lambda} + 2V_\mu\frac{\sqrt{\mu}}{\lambda+\mu}\frac{\partial}{\partial\mu}\left(\sqrt{\lambda+\mu}\,V_\lambda\right) +$$

$$V_\varphi\frac{1}{\sqrt{\lambda\mu}}\frac{\partial V_\lambda}{\partial\varphi} - V_\mu^2\frac{\sqrt{\lambda}}{(\lambda+\mu)^{3/2}} - V_\varphi^2\frac{1}{\sqrt{\lambda(\lambda+\mu)}}\,, \qquad (97)$$

$$\hat{f}_\mu = \frac{\partial V_\mu}{\partial t} + 2V_\lambda\frac{\sqrt{\lambda}}{\lambda+\mu}\frac{\partial}{\partial\lambda}\left(\sqrt{\lambda+\mu}\,V_\mu\right) + 2V_\mu\sqrt{\frac{\mu}{\lambda+\mu}}\,\frac{\partial V_\mu}{\partial\mu} +$$

$$V_\varphi\frac{1}{\sqrt{\lambda\mu}}\frac{\partial V_\mu}{\partial\varphi} - V_\lambda^2\frac{\sqrt{\mu}}{(\lambda+\mu)^{3/2}} - V_\varphi^2\frac{1}{\sqrt{\mu(\lambda+\mu)}}\,, \qquad (98)$$

$$\hat{f}_\varphi = \frac{\partial V_\varphi}{\partial t} + \frac{2}{\sqrt{\lambda+\mu}}\Big[V_\lambda\frac{\partial}{\partial\lambda}\left(\sqrt{\lambda}\,V_\varphi\right) +$$

$$V_\mu\frac{\partial}{\partial\mu}\left(\sqrt{\mu}\,V_\varphi\right)\Big] + \frac{V_\varphi}{\sqrt{\lambda\mu}}\frac{\partial V_\varphi}{\partial\varphi}\,, \qquad (99)$$

$$\frac{g^{jk}}{\sqrt{g_{11}}}\,\tau_{1j,k} = -2\sqrt{\frac{\lambda}{\lambda+\mu}}\,\frac{\partial p}{\partial\lambda} + \frac{2}{\sqrt{\lambda(\lambda+\mu)}}\frac{\partial}{\partial\lambda}\left(\lambda\,\hat{\pi}^{\lambda\lambda}\right) +$$

$$\frac{2}{\sqrt{\lambda+\mu}}\frac{\partial}{\partial\mu}\left(\sqrt{\mu}\,\hat{\pi}^{\lambda\mu}\right) + \frac{1}{\sqrt{\lambda\mu}}\frac{\partial}{\partial\varphi}\left(\hat{\pi}^{\lambda\varphi}\right) -$$

$$\frac{\mu}{\sqrt{\lambda}\,(\lambda+\mu)^{3/2}}\,\hat{\pi}^{\lambda\lambda} - \frac{\sqrt{\lambda}}{(\lambda+\mu)^{3/2}}\,\hat{\pi}^{\mu\mu} -$$

$$\frac{1}{\sqrt{\lambda(\lambda+\mu)}}\,\hat{\pi}^{\varphi\varphi} + \frac{2\sqrt{\mu}}{(\lambda+\mu)^{3/2}}\,\hat{\pi}^{\lambda\mu}\,, \qquad (100)$$

$$\frac{g^{jk}}{\sqrt{g_{22}}}\,\Upsilon_{2i,k} = -2\sqrt{\frac{\mu}{\lambda+\mu}}\frac{\partial p}{\partial \mu} + \frac{2}{\sqrt{\lambda+\mu}}\frac{\partial}{\partial\lambda}\left(\sqrt{\lambda}\,\hat{\Pi}^{\mu\lambda}\right) +$$

$$\frac{2}{\sqrt{\mu(\lambda+\mu)}}\frac{\partial}{\partial\mu}\left(\mu\,\hat{\Pi}^{\mu\mu}\right) + \frac{1}{\sqrt{\lambda\mu}}\frac{\partial}{\partial\varphi}\left(\hat{\Pi}^{\mu\varphi}\right) -$$

$$\frac{\sqrt{\mu}}{(\lambda+\mu)^{3/2}}\,\hat{\Pi}^{\lambda\lambda} - \frac{\lambda}{\sqrt{\mu}\,(\lambda+\mu)^{3/2}}\,\hat{\Pi}^{\mu\mu} -$$

$$\frac{1}{\sqrt{\mu(\lambda+\mu)}}\,\hat{\Pi}^{\varphi\varphi} + \frac{2\sqrt{\lambda}}{(\lambda+\mu)^{3/2}}\,\hat{\Pi}^{\mu\lambda}, \tag{101}$$

and

$$\frac{g^{jk}}{\sqrt{g_{33}}}\,\Upsilon_{3i,k} = -\frac{1}{\sqrt{\lambda\mu}}\frac{\partial p}{\partial\varphi} + \frac{\sqrt{\lambda}}{\lambda+\mu}\frac{\partial}{\partial\lambda}\left(\sqrt{\lambda+\mu}\,\hat{\Pi}^{\varphi\lambda}\right) +$$

$$\frac{\sqrt{\mu}}{\lambda+\mu}\frac{\partial}{\partial\mu}\left(\sqrt{\lambda+\mu}\,\hat{\Pi}^{\varphi\mu}\right) + \frac{1}{\sqrt{\lambda\mu}}\frac{\partial}{\partial\varphi}\,\hat{\Pi}^{\varphi\varphi} +$$

$$\frac{1}{\sqrt{\lambda(\lambda+\mu)}}\,\hat{\Pi}^{\varphi\lambda} + \frac{1}{\sqrt{\mu(\lambda+\mu)}}\,\hat{\Pi}^{\varphi\mu}. \tag{102}$$

<u>Prolate Spheroidal Coordinates</u>

$$x^1 = \lambda \qquad x^2 = \mu \qquad x^3 = \varphi \tag{103}$$

$$g_{11} = a^2\,\frac{\lambda^2-\mu^2}{\lambda^2-1} \tag{104}$$

$$g_{22} = a^2\,\frac{\lambda^2-\mu^2}{1-\mu^2} \tag{105}$$

$$g_{33} = a^2(\lambda^2-1)(1-\mu^2) \tag{106}$$

$$g = a^6(\lambda^2-\mu^2)^2 \tag{107}$$

Continuity Equation.

$$\frac{\partial\rho}{\partial t} + \frac{1}{a(\lambda^2-\mu^2)}\frac{\partial}{\partial\lambda}\left[\sqrt{(\lambda^2-1)(\lambda^2-\mu^2)}\,\rho V_\lambda\right] + \frac{1}{a(\lambda^2-\mu^2)}\frac{\partial}{\partial\mu}\left[\sqrt{(\lambda^2-\mu^2)(1-\mu^2)}\,\rho V_\mu\right] + \frac{1}{a\sqrt{(\lambda^2-1)(1-\mu^2)}}\frac{\partial}{\partial\varphi}\left[\rho V_\varphi\right] = 0 \tag{108}$$

Energy Equation. The left side of equation (5) becomes

$$\rho\left\{\frac{\partial}{\partial t}\left[h + \frac{1}{2}(V_\lambda^2 + V_\mu^2 + V_\varphi^2)\right] + \frac{V_\lambda}{a}\sqrt{\frac{\lambda^2-1}{\lambda^2-\mu^2}}\frac{\partial}{\partial\lambda}\left[h + \frac{1}{2}(V_\lambda^2 + V_\mu^2 + V_\varphi^2)\right] + \frac{V_\mu}{a}\sqrt{\frac{1-\mu^2}{\lambda^2-\mu^2}}\frac{\partial}{\partial\mu}\left[h + \frac{1}{2}(V_\lambda^2 + V_\mu^2 + V_\varphi^2)\right] + \frac{V_\varphi}{a\sqrt{(\lambda^2-1)(1-\mu^2)}}\frac{\partial}{\partial\varphi}\left[h + \frac{1}{2}(V_\lambda^2 + V_\mu^2 + V_\varphi^2)\right]\right\}. \tag{109}$$

Also,

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\frac{\sqrt{g}}{\sqrt{g_{(i)(j)}}}V_i\hat{\pi}^{ij}\right)=\frac{1}{a(\lambda^2-\mu^2)}\frac{\partial}{\partial\lambda}\left[\sqrt{(\lambda^2-\mu^2)(\lambda^2-1)}\left(V_\lambda\hat{\pi}^{\lambda\lambda}+\right.\right.$$

$$\left.\left.V_\mu\hat{\pi}^{\mu\lambda}+V_\varphi\hat{\pi}^{\varphi\lambda}\right)\right]+\frac{1}{a(\lambda^2-\mu^2)}\frac{\partial}{\partial\mu}\left[\sqrt{(\lambda^2-\mu^2)(1-\mu^2)}\left(V_\lambda\hat{\pi}^{\lambda\mu}+\right.\right.$$

$$\left.\left.V_\mu\hat{\pi}^{\mu\mu}+V_\varphi\hat{\pi}^{\varphi\mu}\right)\right]+\frac{1}{a\sqrt{(\lambda^2-1)(1-\mu^2)}}\frac{\partial}{\partial\varphi}\left[V_\lambda\hat{\pi}^{\lambda\varphi}+\right.$$

$$\left.V_\mu\hat{\pi}^{\mu\varphi}+V_\varphi\hat{\pi}^{\varphi\varphi}\right], \tag{110}$$

$$\rho V_i\hat{F}_i=\rho\left[V_\lambda\hat{F}_\lambda+V_\mu\hat{F}_\mu+V_\varphi\hat{F}_\varphi\right], \tag{111}$$

and

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\frac{\sqrt{g}}{\sqrt{g_{(i)(i)}}}\hat{q}_i\right)=\frac{1}{a(\lambda^2-\mu^2)}\frac{\partial}{\partial\lambda}\left(\sqrt{(\lambda^2-\mu^2)(\lambda^2-1)}\,\hat{q}_\lambda\right)+$$

$$\frac{1}{a(\lambda^2-\mu^2)}\frac{\partial}{\partial\mu}\left(\sqrt{(\lambda^2-\mu^2)(1-\mu^2)}\,\hat{q}_\mu\right)+$$

$$\frac{1}{a\sqrt{(\lambda^2-1)(1-\mu^2)}}\frac{\partial}{\partial\varphi}\left(\hat{q}_\varphi\right). \tag{112}$$

The components of the viscous stress tensor are

$$\hat{\pi}^{\lambda\mu}=\hat{\pi}^{\mu\lambda}=\nu\left[\frac{\sqrt{\lambda^2-1}}{a}\frac{\partial}{\partial\lambda}\left(\frac{1}{\sqrt{\lambda^2-\mu^2}}V_\mu\right)+\right.$$

$$\left.\frac{\sqrt{1-\mu^2}}{a}\frac{\partial}{\partial\mu}\left(\frac{1}{\sqrt{\lambda^2-\mu^2}}V_\mu\right)\right], \tag{113}$$

$$\hat{\Pi}^{\lambda\varphi} = \hat{\Pi}^{\varphi\lambda} = \nu\left[\frac{\lambda^2-1}{a\sqrt{\lambda^2-\mu^2}}\frac{\partial}{\partial\lambda}\left(\frac{1}{\sqrt{\lambda^2-1}}V_\varphi\right) + \right.$$

$$\left.\frac{1}{a\sqrt{(1-\mu^2)(\lambda^2-1)}}\frac{\partial V_\lambda}{\partial\varphi}\right], \qquad (114)$$

$$\hat{\Pi}^{\mu\varphi} = \hat{\Pi}^{\varphi\mu} = \nu\left[\frac{1-\mu^2}{a\sqrt{\lambda^2-\mu^2}}\frac{\partial}{\partial\mu}\left(\frac{1}{\sqrt{1-\mu^2}}V_\varphi\right) + \right.$$

$$\left.\frac{1}{a\sqrt{(\lambda^2-1)(1-\mu^2)}}\frac{\partial V_\mu}{\partial\varphi}\right], \qquad (115)$$

$$\hat{\Pi}^{\lambda\lambda} = \left(\eta - \frac{2}{3}\nu\right)\left[\frac{1}{a(\lambda^2-\mu^2)}\frac{\partial}{\partial\lambda}\left(\sqrt{(\lambda^2-\mu^2)(\lambda^2-1)}\,V_\lambda\right) + \right.$$

$$\frac{1}{a(\lambda^2-\mu^2)}\frac{\partial}{\partial\mu}\left(\sqrt{(\lambda^2-1)(1-\mu^2)}\,V_\mu\right) +$$

$$\left.\frac{1}{a\sqrt{(\lambda^2-1)(1-\mu^2)}}\frac{\partial V_\varphi}{\partial\varphi}\right] + \nu\left[\frac{2}{a}\frac{\partial}{\partial\lambda}\left(\sqrt{\frac{\lambda^2-1}{\lambda^2-\mu^2}}\,V_\lambda\right) + \right.$$

$$\left. - \frac{2\lambda(1-\mu^2)}{a(\lambda^2-\mu^2)^{3/2}\sqrt{\lambda^2-1}}V_\lambda - \frac{2\mu\sqrt{1-\mu^2}}{a(\lambda^2-\mu^2)^{3/2}}V_\mu\right], \qquad (116)$$

$$\hat{\Pi}^{\mu\mu} = \left(\eta - \frac{2}{3}\nu\right)\left[\frac{1}{a(\lambda^2-\mu^2)}\frac{\partial}{\partial\lambda}\left(\sqrt{(\lambda^2-\mu^2)(\lambda^2-1)}\,V_\lambda\right) + \right.$$

$$\left.\frac{1}{a(\lambda^2-\mu^2)}\frac{\partial}{\partial\mu}\left(\sqrt{(\lambda^2-\mu^2)(1-\mu^2)}\,V_\mu\right) + \frac{1}{\sqrt{(1-\mu^2)(\lambda^2-1)}}\frac{\partial V_\varphi}{\partial\varphi}\right] +$$

$$\nu\left[\frac{2}{a}\frac{\partial}{\partial\mu}\left(\sqrt{\frac{1-\mu^2}{\lambda^2-\mu^2}}\,V_\mu\right) + \frac{2\lambda}{a}\frac{\sqrt{\lambda^2-1}}{(\lambda^2-\mu^2)^{3/2}}V_\lambda + \right.$$

$$\left.\frac{2\mu}{a}\frac{\lambda^2-1}{\sqrt{1-\mu^2}\,(\lambda^2-\mu^2)^{3/2}}V_\mu\right], \qquad (117)$$

and

$$\hat{\Pi}^{\varphi\varphi} = (\eta - \tfrac{2}{3}\tau)\Big[\frac{1}{a\sqrt{\lambda^2-\mu^2}}\frac{\partial}{\partial\lambda}\left(\sqrt{(\lambda^2-\mu^2)(\lambda^2-1)}\;V_\lambda\right) +$$

$$\frac{1}{a(\lambda^2-\mu^2)}\frac{\partial}{\partial\mu}\left(\sqrt{(\lambda^2-\mu^2)(1-\mu^2)}\;V_\mu\right) +$$

$$\frac{1}{\sqrt{(1-\mu^2)(\lambda^2-1)}}\frac{\partial V_\varphi}{\partial\varphi}\Big] + \tau\Big[\frac{2}{a}\frac{\partial}{\partial\mu}\left(\sqrt{\frac{1-\mu^2}{\lambda^2-\mu^2}}\;V_\mu\right) +$$

$$\frac{2\lambda}{a}\frac{\sqrt{\lambda^2-1}}{(\lambda^2-\mu^2)^{3/2}}V_\lambda + \frac{2\mu}{a}\frac{\lambda^2-1}{\sqrt{1-\mu^2}(\lambda^2-\mu^2)}\;V_\mu\Big] . \tag{118}$$

<u>Equation</u> <u>of</u> <u>Motion</u>. In addition to the components of the viscous stress tensor listed above, the terms of interest are

$$\hat{f}_\lambda = \frac{\partial V_\lambda}{\partial t} + \frac{V_\lambda}{a}\sqrt{\frac{\lambda^2-1}{\lambda^2-\mu^2}}\frac{\partial V_\lambda}{\partial\lambda} + \frac{V_\mu}{a}\frac{\sqrt{(\lambda^2-1)(1-\mu^2)}}{\lambda^2-\mu^2}\frac{\partial}{\partial\mu}\left(\sqrt{\frac{\lambda^2-\mu^2}{\lambda^2-1}}\;V_\lambda\right) +$$

$$\frac{V_\varphi}{a}\frac{1}{\sqrt{(\lambda^2-1)(1-\mu^2)}}\frac{\partial V_\lambda}{\partial\varphi} - V_\mu^2\frac{\lambda}{a}\frac{\sqrt{\lambda^2-1}}{(\lambda^2-\mu^2)^{3/2}} -$$

$$V_\varphi^2\frac{\lambda}{a}\frac{1}{\sqrt{(\lambda^2-\mu^2)(\lambda^2-1)}} , \tag{119}$$

$$\hat{f}_{\mu} = \frac{\partial V_{\mu}}{\partial t} + \frac{V_{\lambda}}{a} \frac{\sqrt{\lambda^2-1}}{\lambda^2-\mu^2} \frac{\partial}{\partial \lambda}\left(\sqrt{\lambda^2-\mu^2}\, V_{\mu}\right) +$$

$$\frac{V_{\mu}}{a}\sqrt{\frac{1-\mu^2}{\lambda^2-1}}\frac{\partial V_{\mu}}{\partial \mu} + \frac{V_{\varphi}}{a\sqrt{(\lambda^2-1)(1-\mu^2)}}\frac{\partial V_{\mu}}{\partial \varphi} +$$

$$\frac{\mu V_{\lambda}^2}{a}\frac{\sqrt{1-\mu^2}}{(\lambda^2-\mu^2)^{3/2}} + \frac{\mu V_{\varphi}^2}{a\sqrt{(\lambda^2-\mu^2)(1-\mu^2)}} \quad , \qquad (120)$$

$$\hat{f}_{\varphi} = \frac{\partial V_{\varphi}}{\partial t} + \frac{V_{\lambda}}{a\sqrt{\lambda^2-\mu^2}}\frac{\partial}{\partial \lambda}\left(\sqrt{\lambda^2-1}\, V_{\varphi}\right) +$$

$$\frac{V_{\mu}}{a\sqrt{\lambda^2-\mu^2}}\frac{\partial}{\partial \mu}\left(\sqrt{1-\mu^2}\, V_{\varphi}\right) +$$

$$\frac{V_{\varphi}}{a\sqrt{(\lambda^2-1)(1-\mu^2)}}\frac{\partial V_{\varphi}}{\partial \varphi} \quad , \qquad (121)$$

$$\frac{g^{jk}}{\sqrt{g_{11}}}\hat{\tau}_{1j,k} = -\frac{1}{a}\sqrt{\frac{\lambda^2-1}{\lambda^2-\mu^2}}\frac{\partial p}{\partial \lambda} + \frac{1}{a\sqrt{(\lambda^2-\mu^2)(\lambda^2-1)}}\frac{\partial}{\partial \lambda}\left[(\lambda^2-1)\,\hat{\pi}^{\lambda\lambda}\right] +$$

$$\frac{1}{a\sqrt{\lambda^2-\mu^2}}\frac{\partial}{\partial \mu}\left(\sqrt{1-\mu^2}\,\hat{\pi}^{\lambda\mu}\right) + \frac{1}{a\sqrt{(1-\mu^2)(\lambda^2-1)}}\frac{\partial}{\partial \varphi}\left(\hat{\pi}^{\lambda\varphi}\right) +$$

$$\frac{\lambda(\mu^2-1)}{a\sqrt{\lambda^2-1}\,(\lambda^2-\mu^2)^{3/2}}\hat{\pi}^{\lambda\lambda} - \frac{\lambda\sqrt{\lambda^2-1}}{a(\lambda^2-\mu^2)^{3/2}}\hat{\pi}^{\mu\mu} -$$

$$\frac{\lambda}{a\sqrt{(\lambda^2-1)(\lambda^2-\mu^2)}}\hat{\pi}^{\varphi\varphi} - \frac{2\mu\sqrt{1-\mu^2}}{(\lambda^2-\mu^2)^{3/2}}\hat{\pi}^{\lambda\mu} \quad , \qquad (122)$$

$$\frac{g^{jk}}{\sqrt{g_{22}}}\tau_{2j,k} = -\frac{1}{a}\sqrt{\frac{1-\mu^2}{\lambda^2-\mu^2}}\frac{\partial P}{\partial \mu} + \frac{1}{a\sqrt{\lambda^2-\mu^2}}\frac{\partial}{\partial \lambda}\left(\sqrt{\lambda^2-1}\,\hat{\pi}^{\mu\lambda}\right) +$$

$$\frac{1}{a\sqrt{(1-\mu^2)(\lambda^2-\mu^2)}}\frac{\partial}{\partial \mu}\left[(1-\mu^2)\hat{\pi}^{\mu\mu}\right] +$$

$$\frac{1}{a\sqrt{(1-\mu^2)(\lambda^2-1)}}\frac{\partial}{\partial \varphi}\left(\hat{\pi}^{\mu\varphi}\right) + \frac{\mu\sqrt{1-\mu^2}}{a(\lambda^2-\mu^2)^{3/2}}\hat{\pi}^{\lambda\lambda} +$$

$$\frac{\mu(\lambda^2-1)}{a\sqrt{1-\mu^2}(\lambda^2-\mu^2)^{3/2}}\hat{\pi}^{\mu\mu} + \frac{\mu}{a\sqrt{(1-\mu^2)(\lambda^2-\mu^2)}}\hat{\pi}^{\varphi\varphi} +$$

$$\frac{2\lambda\sqrt{\lambda^2-1}}{a(\lambda^2-\mu^2)^{3/2}}\hat{\pi}^{\mu\lambda} , \qquad (123)$$

and

$$\frac{g^{jk}}{\sqrt{g_{33}}}\tau_{3j,k} = \frac{1}{a\sqrt{(\lambda^2-1)(1-\mu^2)}}\frac{\partial P}{\partial \varphi} +$$

$$\frac{\sqrt{\lambda^2-1}}{a(\lambda^2-\mu^2)}\frac{\partial}{\partial \lambda}\left(\sqrt{\lambda^2-\mu^2}\,\hat{\pi}^{\varphi\lambda}\right) +$$

$$\frac{\sqrt{1-\mu^2}}{a(\lambda^2-\mu^2)}\frac{\partial}{\partial \mu}\left(\sqrt{\lambda^2-\mu^2}\,\hat{\pi}^{\varphi\mu}\right) +$$

$$\frac{1}{a\sqrt{(\lambda^2-1)(1-\mu^2)}}\frac{\partial}{\partial \varphi}\left(\hat{\pi}^{\varphi\varphi}\right) + \frac{2\lambda}{a\sqrt{(\lambda^2-1)(\lambda^2-\mu^2)}}\hat{\pi}^{\varphi\lambda} -$$

$$\frac{2\mu}{a\sqrt{(1-\mu^2)(\lambda^2-\mu^2)}}\hat{\pi}^{\varphi\mu} . \qquad (124)$$

Spheroidal Coordinates (Oblate Spheroids)

$$x^1 = \lambda \qquad x^2 = \mu \qquad x^3 = \varphi \tag{125}$$

$$g_{11} = \frac{a^2(\lambda^2-\mu^2)}{\lambda^2-1} \tag{126}$$

$$g_{22} = \frac{a^2(\lambda^2-\mu^2)}{1-\mu^2} \tag{127}$$

$$g_{33} = a^2\lambda^2\mu^2 \tag{128}$$

$$g = \frac{a^6\lambda^2\mu^2(\lambda^2-\mu^2)^2}{(\lambda^2-1)(1-\mu^2)} \tag{129}$$

Continuity Equation.

$$\frac{\partial\rho}{\partial t} + \frac{\sqrt{\lambda^2-1}}{a\lambda(\lambda^2-\mu^2)}\frac{\partial}{\partial\lambda}\left(\lambda\sqrt{\lambda^2-\mu^2}\,\rho V_\lambda\right) + \frac{\sqrt{1-\mu^2}}{a\mu(\lambda^2-\mu^2)}\frac{\partial}{\partial\mu}\left(\mu\sqrt{\lambda^2-\mu^2}\,\rho V_\mu\right) + \frac{1}{a\lambda\mu}\frac{\partial}{\partial\varphi}\left(\rho V_\varphi\right) = 0 \tag{130}$$

Energy Equation. The left side of equation (5) becomes

$$\rho\Big\{\frac{\partial}{\partial t}\Big[h+\frac{1}{2}(V_\lambda^2+V_\mu^2+V_\varphi^2)\Big]+\frac{V_\lambda}{a}\sqrt{\frac{\lambda^2-1}{\lambda^2-\mu^2}}\frac{\partial}{\partial\lambda}\Big[h+\frac{1}{2}(V_\lambda^2+V_\mu^2+V_\varphi^2)\Big]+\frac{V_\mu}{a}\sqrt{\frac{1-\mu^2}{\lambda^2-\mu^2}}\frac{\partial}{\partial\mu}\Big[h+\frac{1}{2}(V_\lambda^2+V_\mu^2+V_\varphi^2)\Big]+\frac{V_\varphi}{a\lambda\mu}\frac{\partial}{\partial\varphi}\Big[h+\frac{1}{2}(V_\lambda^2+V_\mu^2+V_\varphi^2)\Big]\Big\}\,. \qquad (131)$$

Also,

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\Big(\frac{\sqrt{g}}{\sqrt{g_{(j)(j)}}}V_i\hat{\pi}^{ij}\Big)=\frac{\sqrt{\lambda^2-1}}{a\lambda(\lambda^2-\mu^2)}\frac{\partial}{\partial\lambda}\Big[\lambda\sqrt{\lambda^2-\mu^2}\,(V_\lambda\hat{\pi}^{\lambda\lambda}+V_\mu\hat{\pi}^{\mu\lambda}+V_\varphi\hat{\pi}^{\varphi\lambda})\Big]+\frac{\sqrt{1-\mu^2}}{a\mu(\lambda^2-\mu^2)}\frac{\partial}{\partial\mu}\Big[\mu\sqrt{\lambda^2-\mu^2}\,(V_\lambda\hat{\pi}^{\lambda\mu}+V_\mu\hat{\pi}^{\mu\mu}+V_\varphi\hat{\pi}^{\varphi\mu})\Big]+\frac{1}{a\lambda\mu}\frac{\partial}{\partial\varphi}\Big[V_\lambda\hat{\pi}^{\lambda\varphi}+V_\mu\hat{\pi}^{\mu\varphi}+V_\varphi\hat{\pi}^{\varphi\varphi}\Big]\,, \qquad (132)$$

$$\rho V_i\hat{F}_i=\rho(V_\lambda\hat{F}_\lambda+V_\mu\hat{F}_\mu+V_\varphi\hat{F}_\varphi)\,, \qquad (133)$$

and

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\frac{\sqrt{g}}{\sqrt{g_{(ixi)}}} \hat{g}_i\right) = \frac{\sqrt{\lambda^2-1}}{a\lambda(\lambda^2-\mu^2)} \frac{\partial}{\partial\lambda}\left(\lambda\sqrt{\lambda^2-\mu^2}\,\hat{g}_\lambda\right) +$$

$$\frac{\sqrt{1-\mu^2}}{a\mu(\lambda^2-\mu^2)} \frac{\partial}{\partial\mu}\left(\mu\sqrt{\lambda^2-\mu^2}\,\hat{g}_\mu\right) +$$

$$\frac{1}{a\lambda\mu} \frac{\partial}{\partial\varphi}\left(\hat{g}_\varphi\right) . \qquad (134)$$

The components of the stress tensor are

$$\hat{\pi}^{\lambda\mu} = \hat{\pi}^{\mu\lambda} = \tau\left[\frac{\sqrt{\lambda^2-1}}{a} \frac{\partial}{\partial\lambda}\left(\frac{1}{\sqrt{\lambda^2-\mu^2}} V_\mu\right) +\right.$$

$$\left.\frac{\sqrt{1-\mu^2}}{a} \frac{\partial}{\partial\mu}\left(\frac{1}{\sqrt{\lambda^2-\mu^2}} V_\lambda\right)\right] , \qquad (135)$$

$$\hat{\pi}^{\lambda\varphi} = \hat{\pi}^{\varphi\lambda} = \tau\left[\frac{\lambda}{a}\sqrt{\frac{\lambda^2-1}{\lambda^2-\mu^2}} \frac{\partial}{\partial\lambda}\left(\frac{1}{\lambda} V_\varphi\right) +\right.$$

$$\left.\frac{1}{a\lambda\mu} \frac{\partial V_\lambda}{\partial\varphi}\right] , \qquad (136)$$

$$\hat{\pi}^{\mu\varphi} = \hat{\pi}^{\varphi\mu} = \tau\left[\frac{\mu}{a}\sqrt{\frac{1-\mu^2}{\lambda^2-\mu^2}} \frac{\partial}{\partial\mu}\left(\frac{1}{\mu} V_\varphi\right) +\right.$$

$$\left.\frac{1}{a\lambda\mu} \frac{\partial V_\mu}{\partial\varphi}\right] , \qquad (137)$$

$$\hat{\pi}^{\lambda\lambda} = (\eta - \tfrac{2}{3}\tau)\Big[\frac{\sqrt{\lambda^2-1}}{a\lambda(\lambda^2-\mu^2)}\frac{\partial}{\partial\lambda}\left(\lambda\sqrt{\lambda^2-\mu^2}\,V_\lambda\right) +$$

$$\frac{\sqrt{1-\mu^2}}{a\mu(\lambda^2-\mu^2)}\frac{\partial}{\partial\mu}\left(\mu\sqrt{\lambda^2-\mu^2}\,V_\mu\right) + \frac{1}{a\lambda\mu}\frac{\partial V_\varphi}{\partial\varphi}\Big] +$$

$$\tau\Big[\frac{2}{a}\frac{\partial}{\partial\lambda}\left(\sqrt{\frac{\lambda^2-1}{\lambda^2-\mu^2}}\,V_\lambda\right) - \frac{2\lambda}{a}\frac{1-\mu^2}{\sqrt{\lambda^2-1}\,(\lambda^2-\mu^2)^{3/2}}V_\lambda -$$

$$\frac{2\mu}{a}\frac{\sqrt{1-\mu^2}}{(\lambda^2-\mu^2)^{3/2}}V_\mu\Big], \qquad (138)$$

$$\hat{\pi}^{\mu\mu} = (\eta - \tfrac{2}{3}\tau)\Big[\frac{\sqrt{\lambda^2-1}}{a\lambda(\lambda^2-\mu^2)}\frac{\partial}{\partial\lambda}\left(\lambda\sqrt{\lambda^2-\mu^2}\,V_\lambda\right) +$$

$$\frac{\sqrt{1-\mu^2}}{a\mu(\lambda^2-\mu^2)}\frac{\partial}{\partial\mu}\left(\mu\sqrt{\lambda^2-\mu^2}\,V_\mu\right) + \frac{1}{a\lambda\mu}\frac{\partial V_\varphi}{\partial\varphi}\Big] +$$

$$\tau\Big[\frac{2}{a}\frac{\partial}{\partial\mu}\left(\sqrt{\frac{1-\mu^2}{\lambda^2-\mu^2}}\,V_\mu\right) + \frac{2\lambda}{a}\frac{\sqrt{\lambda^2-1}}{(\lambda^2-\mu^2)^{3/2}}V_\lambda +$$

$$\frac{2\mu}{a}\frac{\lambda^2-1}{\sqrt{1-\mu^2}\,(\lambda^2-\mu^2)^{3/2}}V_\mu\Big], \qquad (139)$$

and

$$\hat{\pi}^{\varphi\varphi} = (\eta - \tfrac{2}{3}\zeta)\Big[\frac{\sqrt{\lambda^2-1}}{a\lambda(\lambda^2-\mu^2)}\frac{\partial}{\partial\lambda}\left(\lambda\sqrt{\lambda^2-\mu^2}\,V_\lambda\right) +$$

$$\frac{\sqrt{1-\mu^2}}{a\mu(\lambda^2-\mu^2)}\frac{\partial}{\partial\mu}\left(\mu\sqrt{\lambda^2-\mu^2}\,V_\mu\right) + \frac{1}{a\lambda\mu}\frac{\partial V_\varphi}{\partial\varphi}\Big] +$$

$$\zeta\Big[\frac{2}{a\lambda\mu}\frac{\partial V_\varphi}{\partial\varphi} + \frac{2}{a\lambda}\sqrt{\frac{\lambda^2-1}{\lambda^2-\mu^2}}\,V_\lambda +$$

$$\frac{2}{a\mu}\sqrt{\frac{1-\mu^2}{\lambda^2-\mu^2}}\,V_\mu\Big] \; . \qquad (140)$$

<u>Equation of Motion</u>. The important terms in the motion equation are

$$\hat{f}_\lambda = \frac{\partial V_\lambda}{\partial t} + \frac{V_\lambda}{a}\sqrt{\frac{\lambda^2-1}{\lambda^2-\mu^2}}\frac{\partial V_\lambda}{\partial\lambda} +$$

$$\frac{V_\mu}{a}\frac{\sqrt{1-\mu^2}}{\lambda^2-\mu^2}\frac{\partial}{\partial\mu}\left(\sqrt{\lambda^2-\mu^2}\,V_\lambda\right) + \frac{V_\varphi}{a\lambda\mu}\frac{\partial V_\lambda}{\partial\varphi} -$$

$$\frac{V_\mu^2\lambda}{a}\frac{\sqrt{\lambda^2-1}}{(\lambda^2-\mu^2)^{3/2}} - \frac{V_\varphi^2}{a\lambda}\sqrt{\frac{\lambda^2-1}{\lambda^2-\mu^2}} \; , \qquad (141)$$

$$\hat{f}_\mu = \frac{\partial V_\mu}{\partial t} + \frac{V_\lambda}{a}\frac{\sqrt{\lambda^2-1}}{\lambda^2-\mu^2}\frac{\partial}{\partial\lambda}\left(\sqrt{\lambda^2-\mu^2}\,V_\mu\right) +$$

$$\frac{V_\mu}{a}\sqrt{\frac{1-\mu^2}{\lambda^2-\mu^2}}\frac{\partial V_\mu}{\partial\mu} + \frac{V_\varphi}{a\lambda\mu}\frac{\partial V_\mu}{\partial\varphi} +$$

$$\frac{V_\lambda^2\mu}{a}\frac{\sqrt{1-\mu^2}}{(\lambda^2-\mu^2)^{3/2}} - \frac{V_\varphi^2}{a\mu}\sqrt{\frac{1-\mu^2}{\lambda^2-\mu^2}} \; , \qquad (142)$$

$$\hat{f}_{\varphi} = \frac{\partial V_{\varphi}}{\partial t} + \frac{V_{\lambda}}{a\lambda}\sqrt{\frac{\lambda^2-1}{\lambda^2-\mu^2}}\,\frac{\partial}{\partial\lambda}\left(\lambda V_{\varphi}\right) +$$

$$\frac{V_{\mu}}{a\mu}\sqrt{\frac{1-\mu^2}{\lambda^2-\mu^2}}\,\frac{\partial}{\partial\mu}\left(\mu V_{\varphi}\right) + \frac{V_{\varphi}}{a\lambda\mu}\,\frac{\partial V_{\varphi}}{\partial\varphi}\,, \tag{143}$$

$$\frac{g^{jk}}{\sqrt{g_{11}}}\,\tau_{1j,k} = -\frac{1}{a}\sqrt{\frac{\lambda^2-1}{\lambda^2-\mu^2}}\,\frac{\partial p}{\partial\lambda} + \frac{1}{a\lambda\sqrt{\lambda^2-\mu^2}}\,\frac{\partial}{\partial\lambda}\left(\lambda\sqrt{\lambda^2-1}\,\hat{\pi}^{\lambda\lambda}\right) +$$

$$\frac{1}{a\mu}\sqrt{\frac{1-\mu^2}{\lambda^2-\mu^2}}\,\frac{\partial}{\partial\mu}\left(\mu\,\hat{\pi}^{\lambda\mu}\right) + \frac{1}{a\lambda\mu}\,\frac{\partial}{\partial\varphi}\left(\hat{\pi}^{\lambda\varphi}\right) +$$

$$\frac{\lambda(\mu^2-1)}{a\sqrt{\lambda^2-1}\,(\lambda^2-\mu^2)^{3/2}}\,\hat{\pi}^{\lambda\lambda} - \frac{\lambda\sqrt{\lambda^2-1}}{a(\lambda^2-\mu^2)^{3/2}}\,\hat{\pi}^{\mu\mu} -$$

$$\frac{1}{a\lambda}\sqrt{\frac{\lambda^2-1}{\lambda^2-\mu^2}}\,\hat{\pi}^{\varphi\varphi} + \frac{2\mu}{a}\,\frac{\sqrt{1-\mu^2}}{(\lambda^2-\mu^2)^{3/2}}\,\hat{\pi}^{\lambda\mu}\,, \tag{144}$$

$$\frac{g^{jk}}{\sqrt{g_{22}}}\,\tau_{2i,k} = -\frac{1}{a}\sqrt{\frac{1-\mu^2}{\lambda^2-\mu^2}}\,\frac{\partial p}{\partial\mu} + \frac{1}{a\lambda}\sqrt{\frac{\lambda^2-1}{\lambda^2-\mu^2}}\,\frac{\partial}{\partial\lambda}\left(\lambda\,\hat{\pi}^{\mu\lambda}\right) +$$

$$\frac{1}{a\mu\sqrt{\lambda^2-\mu^2}}\,\frac{\partial}{\partial\mu}\left(\mu\sqrt{\lambda^2-\mu^2}\,\hat{\pi}^{\mu\mu}\right) + \frac{1}{a\lambda\mu}\,\frac{\partial}{\partial\varphi}\left(\hat{\pi}^{\mu\varphi}\right) +$$

$$\frac{\mu(\lambda^2-1)}{a\sqrt{1-\mu^2}\,(\lambda^2-\mu^2)^{3/2}}\,\hat{\pi}^{\mu\mu} + \frac{\mu\sqrt{1-\mu^2}}{a(\lambda^2-\mu^2)^{3/2}}\,\hat{\pi}^{\lambda\lambda} -$$

$$\frac{1}{a\mu}\sqrt{\frac{1-\mu^2}{\lambda^2-\mu^2}}\,\hat{\pi}^{\varphi\varphi} + \frac{2\lambda\sqrt{\lambda^2-1}}{a(\lambda^2-\mu^2)^{3/2}}\,\hat{\pi}^{\mu\lambda}\,, \tag{145}$$

and

$$\frac{g^{jk}}{\sqrt{g_{33}}} \Upsilon_{3j,k} = -\frac{1}{a\lambda\mu}\frac{\partial p}{\partial \varphi} + \frac{\sqrt{\lambda^2-1}}{a\lambda(\lambda^2-\mu^2)}\frac{\partial}{\partial\lambda}\left(\sqrt{\lambda^2-\mu^2}\,\hat{\Pi}^{\varphi\lambda}\right) +$$

$$\frac{\sqrt{1-\mu^2}}{a\mu(\lambda^2-\mu^2)}\frac{\partial}{\partial\mu}\left(\sqrt{\lambda^2-\mu^2}\,\hat{\Pi}^{\varphi\mu}\right) + \frac{1}{a\lambda\mu}\frac{\partial}{\partial\varphi}\left(\hat{\Pi}^{\varphi\varphi}\right) +$$

$$\frac{2}{a\lambda}\sqrt{\frac{\lambda^2-1}{\lambda^2-\mu^2}}\,\hat{\Pi}^{\varphi\lambda} + \frac{2}{a\mu}\sqrt{\frac{1-\mu^2}{\lambda^2-\mu^2}}\,\hat{\Pi}^{\varphi\mu}. \tag{146}$$

Parabolic Cylinder Coordinates

$$x^1 = \lambda \qquad x^2 = \mu \qquad x^3 = z \tag{147}$$

$$g_{11} = \frac{\lambda+\mu}{4\lambda} \tag{148}$$

$$g_{22} = \frac{\lambda+\mu}{4\mu} \tag{149}$$

$$g_{33} = 1 \tag{150}$$

$$g = \frac{(\lambda+\mu)^2}{16\lambda\mu} \tag{151}$$

Continuity Equation.

$$\frac{\partial\rho}{\partial t} + \frac{2\sqrt{\lambda}}{\lambda+\mu}\frac{\partial}{\partial\lambda}\left(\sqrt{\lambda+\mu}\,\rho V_\lambda\right) +$$

$$\frac{2\sqrt{\mu}}{\lambda+\mu}\frac{\partial}{\partial\mu}\left(\sqrt{\lambda+\mu}\,\rho V_\mu\right) + \frac{\partial}{\partial z}\left(\rho V_z\right) = 0 \tag{152}$$

<u>Energy</u> <u>Equation</u>. The left side of equation (5) becomes

$$\rho\left\{\frac{\partial}{\partial t}\left[h+\frac{1}{2}(V_\lambda^2+V_\mu^2+V_z^2)\right]+2V_\lambda\sqrt{\frac{\lambda}{\lambda+\mu}}\frac{\partial}{\partial\lambda}\left[h+\frac{1}{2}(V_\lambda^2+V_\mu^2+V_z^2)\right]+2V_\mu\sqrt{\frac{\mu}{\lambda+\mu}}\frac{\partial}{\partial\mu}\left[h+\frac{1}{2}(V_\lambda^2+V_\mu^2+V_z^2)\right]+V_z\frac{\partial}{\partial z}\left[h+\frac{1}{2}(V_\lambda^2+V_\mu^2+V_z^2)\right]\right\}. \tag{153}$$

Also,

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(\frac{\sqrt{g}}{\sqrt{g_{(j)(j)}}}V_i\hat{\pi}^{ij}\right)=\frac{2\sqrt{\lambda}}{\lambda+\mu}\frac{\partial}{\partial\lambda}\left[\sqrt{\lambda+\mu}\left(V_\lambda\hat{\pi}^{\lambda\lambda}+V_\mu\hat{\pi}^{\mu\lambda}+V_z\hat{\pi}^{z\lambda}\right)\right]+\frac{2\sqrt{\mu}}{\lambda+\mu}\frac{\partial}{\partial\mu}\left[\sqrt{\lambda+\mu}\left(V_\lambda\hat{\pi}^{\lambda\mu}+V_\mu\hat{\pi}^{\mu\mu}+V_z\hat{\pi}^{z\mu}\right)\right]+\frac{\partial}{\partial z}\left(V_\lambda\hat{\pi}^{\lambda z}+V_\mu\hat{\pi}^{\mu z}+V_z\hat{\pi}^{zz}\right), \tag{154}$$

$$\rho V_i\hat{F}_i=\rho\left(V_\lambda\hat{F}_\lambda+V_\mu\hat{F}_\mu+V_z\hat{F}_z\right), \tag{155}$$

and

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\frac{\sqrt{g}}{\sqrt{g_{(i)(i)}}}\hat{q}_i\right)=\frac{2\sqrt{\lambda}}{\lambda+\mu}\frac{\partial}{\partial\lambda}\left(\sqrt{\lambda+\mu}\,\hat{q}_\lambda\right)+\frac{2\sqrt{\mu}}{\lambda+\mu}\frac{\partial}{\partial\mu}\left(\sqrt{\lambda+\mu}\,\hat{q}_\mu\right)+\frac{\partial}{\partial z}\left(\hat{q}_z\right). \tag{156}$$

The components of the stress tensor are

$$\hat{\Pi}^{\lambda\mu} = \hat{\Pi}^{\mu\lambda} = \tau\Big[2\sqrt{\lambda}\,\frac{\partial}{\partial\lambda}\Big(\frac{1}{\sqrt{\lambda+\mu}}\,V_\mu\Big) +$$

$$2\sqrt{\mu}\,\frac{\partial}{\partial\mu}\Big(\frac{1}{\sqrt{\lambda+\mu}}\,V_\lambda\Big)\Big] \quad , \qquad (157)$$

$$\hat{\Pi}^{\lambda z} = \hat{\Pi}^{z\lambda} = \tau\Big[2\sqrt{\frac{\lambda}{\lambda+\mu}}\,\frac{\partial V_z}{\partial\lambda} + \frac{\partial V_\lambda}{\partial z}\Big] , \qquad (158)$$

$$\hat{\Pi}^{\mu z} = \hat{\Pi}^{z\mu} = \tau\Big[2\sqrt{\frac{\mu}{\lambda+\mu}}\,\frac{\partial V_z}{\partial\mu} + \frac{\partial V_\mu}{\partial z}\Big] , \qquad (159)$$

$$\hat{\Pi}^{\lambda\lambda} = (\eta - \tfrac{2}{3}\tau)\Big[\frac{2\sqrt{\lambda}}{\lambda+\mu}\,\frac{\partial}{\partial\lambda}\big(\sqrt{\lambda+\mu}\,V_\lambda\big) +$$

$$\frac{2\sqrt{\mu}}{\lambda+\mu}\,\frac{\partial}{\partial\mu}\big(\sqrt{\lambda+\mu}\,V_\mu\big) + \frac{\partial V_z}{\partial z}\Big] + \tau\Big[4\,\frac{\partial}{\partial\lambda}\Big(\sqrt{\frac{\lambda}{\lambda+\mu}}\,V_\lambda\Big) -$$

$$\frac{2\mu}{\sqrt{\lambda}\,(\lambda+\mu)^{3/2}}\,V_\lambda + \frac{2\sqrt{\mu}}{(\lambda+\mu)^{3/2}}\,V_\mu\Big] , \qquad (160)$$

$$\hat{\Pi}^{\mu\mu} = (\eta - \tfrac{2}{3}\tau)\Big[\frac{2\sqrt{\lambda}}{\lambda+\mu}\,\frac{\partial}{\partial\lambda}\big(\sqrt{\lambda+\mu}\,V_\lambda\big) +$$

$$\frac{2\sqrt{\mu}}{\lambda+\mu}\,\frac{\partial}{\partial\mu}\big(\sqrt{\lambda+\mu}\,V_\mu\big) + \frac{\partial V_z}{\partial z}\Big] + \tau\Big[4\,\frac{\partial}{\partial\mu}\Big(\sqrt{\frac{\mu}{\lambda+\mu}}\,V_\mu\Big) +$$

$$\frac{2\sqrt{\lambda}}{(\lambda+\mu)^{3/2}}\,V_\lambda - \frac{2\lambda}{\sqrt{\mu}\,(\lambda+\mu)^{3/2}}\,V_\mu \quad , \qquad (161)$$

and

$$\hat{\Pi}^{zz} = (\eta - \tfrac{2}{3}\nabla)\Big[\frac{2\sqrt{\lambda}}{\lambda+\mu}\frac{\partial}{\partial\lambda}\left(\sqrt{\lambda+\mu}\,V_\lambda\right) + \frac{2\sqrt{\mu}}{\lambda+\mu}\frac{\partial}{\partial\mu}\left(\sqrt{\lambda+\mu}\,V_\mu\right) + \frac{\partial V_z}{\partial z}\Big] + 2\nabla\frac{\partial V_z}{\partial z} \,. \tag{162}$$

Equation of Motion. The important terms in the motion equation are

$$\hat{f}_\lambda = \frac{\partial V_\lambda}{\partial t} + 2V_\lambda\sqrt{\frac{\lambda}{\lambda+\mu}}\frac{\partial V_\lambda}{\partial\lambda} + 2V_\mu\frac{\sqrt{\mu}}{\lambda+\mu}\frac{\partial}{\partial\mu}\left(\sqrt{\lambda+\mu}\,V_\lambda\right) + V_z\frac{\partial V_\lambda}{\partial z} - V_\mu^2\frac{\sqrt{\lambda}}{(\lambda+\mu)^{3/2}} \,, \tag{163}$$

$$\hat{f}_\mu = \frac{\partial V_\mu}{\partial t} + 2V_\lambda\frac{\sqrt{\lambda}}{\lambda+\mu}\frac{\partial}{\partial\lambda}\left(\sqrt{\lambda+\mu}\,V_\mu\right) + 2V_\mu\sqrt{\frac{\mu}{\lambda+\mu}}\frac{\partial V_\mu}{\partial\mu} + V_z\frac{\partial V_\mu}{\partial z} - V_\lambda^2\frac{\sqrt{\mu}}{(\lambda+\mu)^{3/2}} \,, \tag{164}$$

$$\hat{f}_z = \frac{\partial V_z}{\partial t} + 2V_\lambda\sqrt{\frac{\lambda}{\lambda+\mu}}\frac{\partial V_z}{\partial\lambda} + 2V_\mu\sqrt{\frac{\mu}{\lambda+\mu}}\frac{\partial V_z}{\partial\mu} + V_z\frac{\partial V_z}{\partial z} \,, \tag{165}$$

$$\frac{g^{jk}}{\sqrt{g_{11}}}\tau_{1j,k} = -2\sqrt{\frac{\lambda}{\lambda+\mu}}\frac{\partial p}{\partial \lambda} + \frac{2}{\sqrt{\lambda+\mu}}\frac{\partial}{\partial \lambda}\left(\sqrt{\lambda}\,\hat{\pi}^{\lambda\lambda}\right) +$$

$$2\sqrt{\frac{\mu}{\lambda+\mu}}\frac{\partial}{\partial \mu}\left(\hat{\pi}^{\lambda\mu}\right) + \frac{\partial}{\partial z}\left(\hat{\pi}^{\lambda z}\right) - \frac{\mu}{\sqrt{\lambda}(\lambda+\mu)^{3/2}}\hat{\pi}^{\lambda\lambda} -$$

$$\frac{\sqrt{\lambda}}{(\lambda+\mu)^{3/2}}\hat{\pi}^{\mu\mu} + \frac{2\sqrt{\mu}}{(\lambda+\mu)^{3/2}}\hat{\pi}^{\lambda\mu}, \qquad (166)$$

$$\frac{g^{jk}}{\sqrt{g_{22}}}\tau_{2j,k} = -2\sqrt{\frac{\mu}{\lambda+\mu}}\frac{\partial p}{\partial \mu} + 2\sqrt{\frac{\lambda}{\lambda+\mu}}\frac{\partial}{\partial \lambda}\left(\hat{\pi}^{\mu\lambda}\right) +$$

$$\frac{2}{\sqrt{\lambda+\mu}}\frac{\partial}{\partial \mu}\left(\sqrt{\mu}\,\hat{\pi}^{\mu\mu}\right) + \frac{\partial}{\partial z}\left(\hat{\pi}^{\lambda z}\right) - \frac{\sqrt{\mu}}{(\lambda+\mu)^{3/2}}\hat{\pi}^{\lambda\lambda} -$$

$$\frac{\lambda}{\sqrt{\mu}(\lambda+\mu)^{3/2}}\hat{\pi}^{\mu\mu} + \frac{2\sqrt{\lambda}}{(\lambda+\mu)^{3/2}}\hat{\pi}^{\mu\lambda}, \qquad (167)$$

and

$$\frac{g^{jk}}{\sqrt{g_{33}}}\tau_{3j,k} = -\frac{\partial p}{\partial z} + \frac{2\sqrt{\lambda}}{\lambda+\mu}\frac{\partial}{\partial \lambda}\left(\sqrt{\lambda+\mu}\,\hat{\pi}^{z\lambda}\right) +$$

$$\frac{2\sqrt{\mu}}{\lambda+\mu}\frac{\partial}{\partial \mu}\left(\sqrt{\lambda+\mu}\,\hat{\pi}^{z\mu}\right) + \frac{\partial}{\partial z}\left(\pi^{zz}\right). \qquad (168)$$

Elliptic Cylinder Coordinates

$$x^1 = \lambda \qquad x^2 = \mu \qquad x^3 = z \qquad (169)$$

$$g_{11} = \frac{a^2(\lambda^2-\mu^2)}{\lambda^2-1} \qquad (170)$$

$$g_{22} = \frac{a^2(\lambda^2-\mu^2)}{1-\mu^2} \tag{171}$$

$$g_{33} = 1 \tag{172}$$

$$g = \frac{a^4(\lambda^2-\mu^2)^2}{(\lambda^2-1)(1-\mu^2)} \tag{173}$$

Continuity Equation.

$$\frac{\partial \rho}{\partial t} + \frac{\sqrt{\lambda^2-1}}{a(\lambda^2-\mu^2)}\frac{\partial}{\partial \lambda}\left(\sqrt{\lambda^2-\mu^2}\,\rho V_\lambda\right) + \frac{\sqrt{1-\mu^2}}{a(\lambda^2-\mu^2)}\frac{\partial}{\partial \mu}\left(\sqrt{\lambda^2-\mu^2}\,\rho V_\mu\right) + \frac{\partial}{\partial z}\left(\rho V_z\right) = 0 \tag{174}$$

Energy Equation. The left side of equation (5) becomes

$$\rho\Big\{\frac{\partial}{\partial t}\left[h + \frac{1}{2}(V_\lambda^2 + V_\mu^2 + V_z^2)\right] + \frac{V_\lambda}{a}\sqrt{\frac{\lambda^2-1}{\lambda^2-\mu^2}}\frac{\partial}{\partial \lambda}\left[h + \frac{1}{2}(V_\lambda^2 + V_\mu^2 + V_z^2)\right] + \frac{V_\mu}{a}\sqrt{\frac{1-\mu^2}{\lambda^2-\mu^2}}\frac{\partial}{\partial \mu}\left[h + \frac{1}{2}(V_\lambda^2 + V_\mu^2 + V_z^2)\right] + V_z\frac{\partial}{\partial z}\left[h + \frac{1}{2}(V_\lambda^2 + V_\mu^2 + V_z^2)\right]\Big\}. \tag{175}$$

Also,

$$\rho V_i \hat{F}_i = \rho \left[V_\lambda \hat{F}_\lambda + V_\mu \hat{F}_\mu + V_z \hat{F}_z \right], \tag{176}$$

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j}\left(\frac{\sqrt{g}}{\sqrt{g_{(j)(j)}}} V_i \hat{\pi}^{ij}\right) = \frac{\sqrt{\lambda^2-1}}{a(\lambda^2-\mu^2)} \frac{\partial}{\partial\lambda}\left[\sqrt{\lambda^2-\mu^2}\left(V_\lambda \hat{\pi}^{\lambda\lambda} + V_\mu \hat{\pi}^{\mu\lambda} + V_z \hat{\pi}^{z\lambda}\right)\right] + \frac{\sqrt{1-\mu^2}}{a(\lambda^2-\mu^2)} \frac{\partial}{\partial\mu}\left[\sqrt{\lambda^2-\mu^2}\left(V_\lambda \hat{\pi}^{\lambda\mu} + V_\mu \hat{\pi}^{\mu\mu} + V_z \hat{\pi}^{\mu z}\right)\right] + \frac{\partial}{\partial z}\left[V_\lambda \hat{\pi}^{\lambda z} + V_\mu \hat{\pi}^{\mu z} + V_z \hat{\pi}^{zz}\right], \tag{177}$$

and

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\frac{\sqrt{g}}{\sqrt{g_{(i)(i)}}} \hat{q}_i\right) = \frac{\sqrt{\lambda^2-1}}{a(\lambda^2-\mu^2)} \frac{\partial}{\partial\lambda}\left(\sqrt{\lambda^2-\mu^2}\, \hat{q}_\lambda\right) + \frac{\sqrt{1-\mu^2}}{a(\lambda^2-\mu^2)} \frac{\partial}{\partial\mu}\left(\sqrt{\lambda^2-\mu^2}\, \hat{q}_\mu\right) + \frac{\partial}{\partial z}\left(\hat{q}_z\right) . \tag{178}$$

The components of the stress tensor are

$$\hat{\pi}^{\lambda\mu} = \hat{\pi}^{\mu\lambda} = \nabla\left[\frac{\sqrt{\lambda^2-1}}{a} \frac{\partial}{\partial\lambda}\left(\frac{1}{\sqrt{\lambda^2-\mu^2}} V_\mu\right) + \frac{\sqrt{1-\mu^2}}{a} \frac{\partial}{\partial\mu}\left(\frac{1}{\sqrt{\lambda^2-\mu^2}} V_\lambda\right)\right], \tag{179}$$

$$\hat{\pi}^{\lambda z} = \hat{\pi}^{z\lambda} = \nabla\left[\frac{1}{a}\sqrt{\frac{\lambda^2-1}{\lambda^2-\mu^2}} \frac{\partial V_z}{\partial\lambda} + \frac{\partial V_\lambda}{\partial z}\right], \tag{180}$$

$$\hat{\pi}^{\mu z} = \hat{\pi}^{z\mu} = \nabla\left[\frac{1}{a}\sqrt{\frac{1-\mu^2}{\lambda^2-\mu^2}} \frac{\partial V_z}{\partial\mu} + \frac{\partial V_\mu}{\partial z}\right], \tag{181}$$

$$\hat{\Pi}^{\lambda\lambda} = \left(\eta - \tfrac{2}{3}\tau\right)\left[\frac{\sqrt{\lambda^2-1}}{a(\lambda^2-\mu^2)}\frac{\partial}{\partial\lambda}\left(\sqrt{\lambda^2-\mu^2}\,V_\lambda\right) + \right.$$

$$\left.\frac{\sqrt{1-\mu^2}}{a(\lambda^2-\mu^2)}\frac{\partial}{\partial\mu}\left(\sqrt{\lambda^2-\mu^2}\,V_\mu\right) + \frac{\partial V_z}{\partial z}\right] + \tau\left[\frac{2}{a}\frac{\partial}{\partial\lambda}\left(\sqrt{\frac{\lambda^2-1}{\lambda^2-\mu^2}}\,V_\lambda\right) - \right.$$

$$\left.\frac{2\lambda(1-\mu^2)}{a\sqrt{\lambda^2-1}\,(\lambda^2-\mu^2)^{3/2}}V_\lambda - \frac{2\mu\sqrt{1-\mu^2}}{a(\lambda^2-\mu^2)^{3/2}}V_\mu\right], \qquad (182)$$

$$\hat{\Pi}^{\mu\mu} = \left(\eta - \tfrac{2}{3}\tau\right)\left[\frac{\sqrt{\lambda^2-1}}{a(\lambda^2-\mu^2)}\frac{\partial}{\partial\lambda}\left(\sqrt{\lambda^2-\mu^2}\,V_\lambda\right) + \right.$$

$$\left.\frac{\sqrt{1-\mu^2}}{a(\lambda^2-\mu^2)}\frac{\partial}{\partial\mu}\left(\sqrt{\lambda^2-\mu^2}\,V_\mu\right) + \frac{\partial V_z}{\partial z}\right] + \tau\left[\frac{2}{a}\frac{\partial}{\partial\mu}\left(\sqrt{\frac{1-\mu^2}{\lambda^2-\mu^2}}\,V_\mu\right) + \right.$$

$$\left.\frac{2\lambda}{a}\frac{\sqrt{\lambda^2-1}}{(\lambda^2-\mu^2)^{3/2}}V_\lambda + \frac{2\mu(\lambda^2-1)}{a\sqrt{1-\mu^2}\,(\lambda^2-\mu^2)^{3/2}}V_\mu\right], \qquad (183)$$

and

$$\hat{\Pi}^{zz} = \left(\eta - \tfrac{2}{3}\tau\right)\left[\frac{\sqrt{\lambda^2-1}}{a(\lambda^2-\mu^2)}\frac{\partial}{\partial\lambda}\left(\sqrt{\lambda^2-\mu^2}\,V_\lambda\right) + \right.$$

$$\left.\frac{\sqrt{1-\mu^2}}{a(\lambda^2-\mu^2)}\frac{\partial}{\partial\mu}\left(\sqrt{\lambda^2-\mu^2}\,V_\mu\right) + \frac{\partial V_z}{\partial z}\right] + 2\tau\frac{\partial V_z}{\partial z}. \qquad (184)$$

Equation of Motion. In addition to the components of the stress tensor listed above, the important terms in the equation of motion are

$$\hat{f}_\lambda = \frac{\partial V_\lambda}{\partial t} + \frac{V_\lambda}{a}\sqrt{\frac{\lambda^2-1}{\lambda^2-\mu^2}}\frac{\partial V_\lambda}{\partial \lambda} + \frac{V_\mu}{a}\frac{\sqrt{1-\mu^2}}{\lambda^2-\mu^2}\frac{\partial}{\partial \mu}\left(\sqrt{\lambda^2-\mu^2}\,V_\lambda\right) +$$

$$V_z\frac{\partial V_z}{\partial z} - \frac{V_\mu^{\,2}\lambda}{a}\frac{\sqrt{\lambda^2-1}}{(\lambda^2-\mu^2)^{3/2}} \quad , \qquad (185)$$

$$\hat{f}_\mu = \frac{\partial V_\mu}{\partial t} + \frac{V_\lambda}{a}\frac{\sqrt{\lambda^2-1}}{\lambda^2-\mu^2}\frac{\partial}{\partial \lambda}\left(\sqrt{\lambda^2-\mu^2}\,V_\mu\right) + \frac{V_\mu}{a}\sqrt{\frac{1-\mu^2}{\lambda^2-\mu^2}}\frac{\partial V_\mu}{\partial \mu} +$$

$$V_z\frac{\partial V_\mu}{\partial z} + \frac{V_\lambda^{\,2}\mu}{a}\frac{\sqrt{1-\mu^2}}{(\lambda^2-\mu^2)^{3/2}} \quad , \qquad (186)$$

$$\hat{f}_z = \frac{\partial V_z}{\partial t} + \frac{V_\lambda}{a}\sqrt{\frac{\lambda^2-1}{\lambda^2-\mu^2}}\frac{\partial V_z}{\partial \lambda} +$$

$$\frac{V_\mu}{a}\sqrt{\frac{1-\mu^2}{\lambda^2-\mu^2}}\frac{\partial V_z}{\partial \mu} + V_z\frac{\partial V_z}{\partial z} \quad , \qquad (187)$$

$$\frac{g^{jk}}{\sqrt{g_{11}}}\tau_{1j,k} = -\frac{1}{a}\sqrt{\frac{\lambda^2-1}{\lambda^2-\mu^2}}\frac{\partial p}{\partial \lambda} + \frac{1}{a\sqrt{\lambda^2-\mu^2}}\frac{\partial}{\partial \lambda}\left(\sqrt{\lambda^2-1}\,\hat{\pi}^{\lambda\lambda}\right) +$$

$$\frac{1}{a}\sqrt{\frac{1-\mu^2}{\lambda^2-\mu^2}}\frac{\partial}{\partial \mu}\left(\hat{\pi}^{\lambda\mu}\right) + \frac{\partial}{\partial z}\left(\hat{\pi}^{\lambda z}\right) + \frac{\lambda(\mu^2-1)}{a\sqrt{\lambda^2-1}\,(\lambda^2-\mu^2)^{3/2}}\hat{\pi}^{\lambda\lambda} -$$

$$\frac{\lambda\sqrt{\lambda^2-1}}{a(\lambda^2-\mu^2)^{3/2}}\hat{\pi}^{\mu\mu} - \frac{2\mu\sqrt{1-\mu^2}}{a(\lambda^2-\mu^2)^{3/2}}\hat{\pi}^{\lambda\mu} \quad , \qquad (188)$$

$$\frac{g^{jk}}{\sqrt{g_{22}}}\tau_{2j,k} = -\frac{1}{a}\sqrt{\frac{1-\mu^2}{\lambda^2-\mu^2}}\frac{\partial p}{\partial \mu} + \frac{1}{a}\sqrt{\frac{\lambda^2-1}{\lambda^2-\mu^2}}\frac{\partial}{\partial \lambda}\left(\hat{\pi}^{\mu\lambda}\right) +$$

$$\frac{1}{a\sqrt{\lambda^2-\mu^2}}\frac{\partial}{\partial \mu}\left(\sqrt{1-\mu^2}\,\hat{\pi}^{\mu\mu}\right) + \frac{\partial}{\partial z}\left(\hat{\pi}^{\mu z}\right) + \frac{\mu\sqrt{1-\mu^2}}{a(\lambda^2-\mu^2)^{3/2}}\hat{\pi}^{\lambda\lambda} +$$

$$\frac{\mu(\lambda^2-1)}{a\sqrt{1-\mu^2}\,(\lambda^2-\mu^2)^{3/2}}\hat{\pi}^{\mu\mu} + \frac{2\lambda\sqrt{\lambda^2-1}}{a(\lambda^2-\mu^2)^{3/2}}\hat{\pi}^{\mu\lambda} \quad , \qquad (189)$$

and

$$\frac{g^{jk}}{\sqrt{g_{33}}}\,\tau_{3j,k} = -\frac{\partial p}{\partial z} + \frac{\sqrt{\lambda^2-1}}{a(\lambda^2-\mu^2)}\frac{\partial}{\partial \lambda}\left(\sqrt{\lambda^2-\mu^2}\,\hat{\tau}^{z\lambda}\right) +$$

$$\frac{\sqrt{1-\mu^2}}{a(\lambda^2-\mu^2)}\frac{\partial}{\partial \mu}\left(\sqrt{\lambda^2-\mu^2}\,\hat{\tau}^{z\mu}\right) + \frac{\partial}{\partial z}\left(\hat{\tau}^{zz}\right) \, . \qquad (190)$$

<u>Ellipsoidal</u> <u>Coordinates</u>

$$X^1 = \lambda \qquad X^2 = \mu \qquad X^3 = \nu \qquad (191)$$

$$g_{11} = \frac{(\mu-\lambda)(\nu-\lambda)}{4(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)} \qquad (192)$$

$$g_{22} = \frac{(\lambda-\mu)(\nu-\mu)}{4(a^2-\mu)(b^2-\mu)(c^2-\mu)} \qquad (193)$$

$$g_{33} = \frac{(\lambda-\nu)(\mu-\nu)}{4(a^2-\nu)(b^2-\nu)(c^2-\nu)} \qquad (194)$$

$$g = \frac{(\mu-\lambda)^2(\nu-\lambda)^2(\nu-\mu)^2}{64(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)(a^2-\mu)(b^2-\mu)(\mu-c^2)(a^2-\nu)(b^2-\nu)(c^2-\nu)} \qquad (195)$$

<u>Continuity</u> <u>Equation</u>.

$$\frac{\partial \rho}{\partial t} + \frac{2\sqrt{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}}{(\lambda-\mu)(\lambda-\nu)} \frac{\partial}{\partial \lambda}\left(\sqrt{(\lambda-\mu)(\lambda-\nu)}\,\rho V_\lambda\right) +$$

$$\frac{2\sqrt{(a^2-\mu)(b^2-\mu)(\mu-c^2)}}{(\mu-\lambda)(\nu-\mu)} \frac{\partial}{\partial \mu}\left(\sqrt{(\mu-\lambda)(\nu-\mu)}\,\rho V_\mu\right) +$$

$$\frac{2\sqrt{(a^2-\nu)(b^2-\nu)(c^2-\nu)}}{(\nu-\lambda)(\nu-\mu)} \frac{\partial}{\partial \nu}\left(\sqrt{(\nu-\lambda)(\nu-\mu)}\,\rho V_\nu\right) = 0 \qquad (196)$$

<u>Energy</u> <u>Equation</u>. The left side of equation (5) becomes

$$\rho\left\{\frac{\partial}{\partial t}\left[h + \frac{1}{2}(V_\lambda^2 + V_\mu^2 + V_\nu^2)\right] +\right.$$

$$2V_\lambda\sqrt{\frac{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}{(\mu-\lambda)(\nu-\lambda)}} \frac{\partial}{\partial \lambda}\left[h + \frac{1}{2}(V_\lambda^2 + V_\mu^2 + V_\nu^2)\right] +$$

$$2V_\mu\sqrt{\frac{(a^2-\mu)(b^2-\mu)(\mu-c^2)}{(\mu-\lambda)(\nu-\mu)}} \frac{\partial}{\partial \mu}\left[h + \frac{1}{2}(V_\lambda^2 + V_\mu^2 + V_\nu^2)\right] +$$

$$\left.2V_\nu\sqrt{\frac{(a^2-\nu)(b^2-\nu)(c^2-\nu)}{(\lambda-\nu)(\mu-\nu)}} \frac{\partial}{\partial \nu}\left[h + \frac{1}{2}(V_\lambda^2 + V_\mu^2 + V_\nu^2)\right]\right\} . \qquad (197)$$

Also,

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(\frac{\sqrt{g}}{\sqrt{g_{(j)(j)}}}V_i\hat{\pi}^{ij}\right) = \frac{2\sqrt{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}}{(\lambda-\mu)(\lambda-\nu)}\frac{\partial}{\partial\lambda}\left[\sqrt{(\lambda-\mu)(\lambda-\nu)}\left(V_\lambda\hat{\pi}^{\lambda\lambda}+\right.\right.$$

$$\left.\left.V_\mu\hat{\pi}^{\mu\lambda}+V_\nu\hat{\pi}^{\nu\lambda}\right)\right]+\frac{2\sqrt{(a^2-\mu)(b^2-\mu)(\mu-c^2)}}{(\mu-\lambda)(\nu-\mu)}\frac{\partial}{\partial\mu}\left[\sqrt{(\mu-\lambda)(\nu-\mu)}\left(V_\lambda\hat{\pi}^{\lambda\mu}+\right.\right.$$

$$\left.\left.V_\mu\hat{\pi}^{\mu\mu}+V_\nu\hat{\pi}^{\nu\mu}\right)\right]+\frac{2\sqrt{(a^2-\nu)(b^2-\nu)(c^2-\nu)}}{(\nu-\lambda)(\nu-\mu)}\frac{\partial}{\partial\nu}\left[\sqrt{(\nu-\lambda)(\nu-\mu)}\left(V_\lambda\hat{\pi}^{\lambda\nu}+\right.\right.$$

$$\left.\left.V_\mu\hat{\pi}^{\mu\nu}+V_\nu\hat{\pi}^{\nu\nu}\right)\right], \tag{198}$$

$$\rho V_i\hat{F}_i = \rho\left[V_\lambda\hat{F}_\lambda + V_\mu\hat{F}_\mu + V_\nu\hat{F}_\nu\right], \tag{199}$$

and

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\frac{\sqrt{g}}{\sqrt{g_{(i)(i)}}}\hat{g}_i\right) = \frac{2\sqrt{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}}{(\lambda-\mu)(\lambda-\nu)}\frac{\partial}{\partial\lambda}\left(\sqrt{(\lambda-\mu)(\lambda-\nu)}\,\hat{g}_\lambda\right)+$$

$$\frac{2\sqrt{(a^2-\mu)(b^2-\mu)(\mu-c^2)}}{(\mu-\lambda)(\nu-\mu)}\frac{\partial}{\partial\mu}\left(\sqrt{(\mu-\lambda)(\nu-\mu)}\,\hat{g}_\mu\right)+$$

$$\frac{2\sqrt{(a^2-\nu)(b^2-\nu)(c^2-\nu)}}{(\nu-\lambda)(\nu-\mu)}\frac{\partial}{\partial\nu}\left(\sqrt{(\nu-\lambda)(\nu-\mu)}\,\hat{g}_\nu\right). \tag{200}$$

The components of the stress tensor are

$$\hat{\pi}^{\lambda\mu}=\hat{\pi}^{\mu\lambda}=\sigma\left[2\sqrt{\frac{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}{\mu-\lambda}}\frac{\partial}{\partial\lambda}\left(\frac{1}{\sqrt{\mu-\lambda}}V_\mu\right)+\right.$$

$$\left.2\sqrt{\frac{(a^2-\mu)(b^2-\mu)(\mu-c^2)}{\nu-\mu}}\frac{\partial}{\partial\mu}\left(\frac{1}{\sqrt{\mu-\lambda}}V_\lambda\right)\right], \tag{201}$$

$$\hat{\Pi}^{\lambda\nu} = \hat{\Pi}^{\nu\lambda} = \sigma \left[2\sqrt{\frac{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}{\mu-\lambda}} \frac{\partial}{\partial\lambda}\left(\frac{1}{\sqrt{\nu-\lambda}} V_\nu\right) + \right.$$

$$2\sqrt{\frac{(a^2-\nu)(b^2-\nu)(c^2-\nu)}{\nu-\mu}} \frac{\partial}{\partial\nu}\left(\frac{1}{\sqrt{\nu-\lambda}} V_\lambda\right) , \tag{202}$$

$$\hat{\Pi}^{\mu\nu} = \hat{\Pi}^{\nu\mu} = \sigma \left[2\sqrt{\frac{(a^2-\mu)(b^2-\mu)(\mu-c^2)}{\mu-\lambda}} \frac{\partial}{\partial\mu}\left(\frac{1}{\sqrt{\nu-\mu}} V_\nu\right) + \right.$$

$$2\sqrt{\frac{(a^2-\nu)(b^2-\nu)(c^2-\nu)}{\nu-\lambda}} \frac{\partial}{\partial\nu}\left(\frac{1}{\sqrt{\nu-\mu}} V_\mu\right) , \tag{203}$$

$$\hat{\Pi}^{\lambda\lambda} = \left(\eta - \frac{2}{3}\sigma\right)\left[\frac{2\sqrt{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}}{(\mu-\lambda)(\nu-\lambda)} \frac{\partial}{\partial\lambda}\left(\sqrt{(\lambda-\mu)(\lambda-\nu)}\, V_\lambda\right) + \right.$$

$$\frac{2\sqrt{(a^2-\mu)(b^2-\mu)(\mu-c^2)}}{(\mu-\lambda)(\mu-\nu)} \frac{\partial}{\partial\mu}\left(\sqrt{(\mu-\lambda)(\nu-\mu)}\, V_\mu\right) +$$

$$\left. \frac{2\sqrt{(a^2-\nu)(b^2-\nu)(c^2-\nu)}}{(\nu-\lambda)(\nu-\mu)} \frac{\partial}{\partial\nu}\left(\sqrt{(\nu-\lambda)(\nu-\mu)}\, V_\nu\right)\right] +$$

$$\sigma \left[4 \frac{\partial}{\partial\lambda}\left(\sqrt{\frac{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}{(\mu-\lambda)(\nu-\lambda)}}\, V_\lambda\right) + \right.$$

$$\frac{2\sqrt{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}\,(2\lambda-\mu-\nu)}{(\mu-\lambda)^{3/2}(\nu-\lambda)^{3/2}} V_\lambda +$$

$$\frac{2\left\{(a^2-\lambda)(b^2-\lambda) + (a^2-\lambda)(c^2-\lambda) + (b^2-\lambda)(c^2-\lambda)\right\}}{\sqrt{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)(\mu-\lambda)(\nu-\lambda)}} V_\lambda +$$

$$\frac{2}{\mu-\lambda}\sqrt{\frac{(a^2-\mu)(b^2-\mu)(c^2-\mu)}{(\lambda-\mu)(\nu-\mu)}}\, V_\mu +$$

$$\left. \frac{2}{\nu-\lambda}\sqrt{\frac{(a^2-\nu)(b^2-\nu)(c^2-\nu)}{(\lambda-\nu)(\mu-\nu)}}\, V_\nu \right] , \tag{204}$$

$$\hat{\Pi}^{\mu\mu} = \left(\eta - \tfrac{2}{3}\tau\right)\Bigg[\frac{2\sqrt{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}}{(\mu-\lambda)(\nu-\lambda)} \frac{\partial}{\partial\lambda}\left(\sqrt{(\mu-\lambda)(\nu-\lambda)}\, V_\lambda\right) +$$

$$\frac{2\sqrt{(a^2-\mu)(b^2-\mu)(\mu-c^2)}}{(\mu-\lambda)(\nu-\mu)} \frac{\partial}{\partial\mu}\left(\sqrt{(\mu-\lambda)(\nu-\mu)}\, V_\mu\right) +$$

$$\frac{2\sqrt{(a^2-\nu)(b^2-\nu)(c^2-\nu)}}{(\nu-\lambda)(\nu-\mu)} \frac{\partial}{\partial\nu}\left(\sqrt{(\nu-\lambda)(\nu-\mu)}\, V_\nu\right)\Bigg] +$$

$$\tau\Bigg[4\frac{\partial}{\partial\mu}\left(\sqrt{\frac{(a^2-\mu)(b^2-\mu)(c^2-\mu)}{(\lambda-\mu)(\nu-\mu)}}\, V_\mu\right) +$$

$$\frac{2}{\lambda-\mu}\sqrt{\frac{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}{(\mu-\lambda)(\nu-\lambda)}}\; V_\lambda \quad +$$

$$\frac{2\sqrt{(a^2-\mu)(b^2-\mu)(\mu-c^2)}}{(\mu-\lambda)^{3/2}(\nu-\mu)^{3/2}}\; V_\mu\,(2\mu-\lambda-\nu) \quad +$$

$$\frac{2\left\{(a^2-\mu)(b^2-\mu) + (a^2-\mu)(c^2-\mu) + (b^2-\mu)(c^2-\mu)\right\}}{\sqrt{(a^2-\mu)(b^2-\mu)(c^2-\mu)(\lambda-\mu)(\nu-\mu)}}\; V_\mu +$$

$$\frac{2}{\lambda-\mu}\sqrt{\frac{(a^2-\nu)(b^2-\nu)(c^2-\nu)}{(\lambda-\nu)(\mu-\nu)}}\quad V_\nu\Bigg] \;, \qquad (205)$$

and

$$\hat{\Pi}^{\nu\nu} = \left(\eta - \tfrac{2}{3}\tau\right)\Bigg[\frac{2\sqrt{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}}{(\nu-\lambda)(\mu-\lambda)} \frac{\partial}{\partial\lambda}\left(\sqrt{(\mu-\lambda)(\nu-\lambda)}\, V_\lambda\right) +$$

$$\frac{2\sqrt{(a^2-\mu)(b^2-\mu)(\mu-c^2)}}{(\nu-\mu)(\mu-\lambda)} \frac{\partial}{\partial\mu}\left(\sqrt{(\mu-\lambda)(\nu-\mu)}\, V_\mu\right) +$$

$$\frac{2\sqrt{(a^2-\nu)(b^2-\nu)(c^2-\nu)}}{(\lambda-\nu)(\mu-\nu)} \frac{\partial}{\partial\nu}\left(\sqrt{(\nu-\lambda)(\nu-\mu)}\, V_\nu\right)\Bigg] +$$

$$\tau \Bigg[4 \frac{\partial}{\partial\nu}\left(\sqrt{\frac{(a^2-\nu)(b^2-\nu)(c^2-\nu)}{(\lambda-\nu)(\mu-\nu)}}\, V_\nu\right) +$$

$$\frac{2}{\lambda-\nu}\sqrt{\frac{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}{(\mu-\lambda)(\nu-\lambda)}}\, V_\lambda +$$

$$\frac{2}{\mu-\nu}\sqrt{\frac{(a^2-\mu)(b^2-\mu)(c^2-\mu)}{(\lambda-\mu)(\nu-\mu)}}\, V_\mu +$$

$$\frac{2\sqrt{(a^2-\nu)(b^2-\nu)(c^2-\nu)}\,(2\nu-\lambda-\mu)}{(\nu-\lambda)^{3/2}(\nu-\mu)^{3/2}}\, V_\nu +$$

$$\frac{2\left\{(a^2-\nu)(b^2-\nu) + (a^2-\nu)(c^2-\nu) + (b^2-\nu)(c^2-\nu)\right\}}{\sqrt{(a^2-\nu)(b^2-\nu)(c^2-\nu)(\lambda-\nu)(\mu-\nu)}}\, V_\nu \Bigg]. \qquad (206)$$

Equation of Motion. The important terms, in addition to the components of the stress tensor listed above, are

$$\hat{f}_\lambda = \frac{\partial V_\lambda}{\partial t} + 2V_\lambda \sqrt{\frac{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}{(\mu-\lambda)(\nu-\lambda)}}\ \frac{\partial V_\lambda}{\partial \lambda} +$$

$$\frac{2V_\mu}{\mu-\lambda} \sqrt{\frac{(a^2-\mu)(b^2-\mu)(c^2-\mu)}{\mu-\nu}}\ \frac{\partial}{\partial \mu}\left(\sqrt{\mu-\lambda}\ V_\lambda\right) +$$

$$\frac{2V_\nu}{\nu-\lambda} \sqrt{\frac{(a^2-\nu)(b^2-\nu)(c^2-\nu)}{\nu-\mu}}\ \frac{\partial}{\partial \nu}\left(\sqrt{\nu-\lambda}\ V_\lambda\right) -$$

$$\frac{V_\mu^2}{\lambda-\mu} \sqrt{\frac{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}{(\mu-\lambda)(\nu-\lambda)}} - \frac{V_\nu^2}{\lambda-\nu} \sqrt{\frac{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}{(\mu-\lambda)(\nu-\lambda)}}\ , \qquad (207)$$

$$\hat{f}_\mu = \frac{\partial V_\mu}{\partial t} + \frac{2V_\lambda}{\lambda-\mu} \sqrt{\frac{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}{\nu-\lambda}}\ \frac{\partial}{\partial \lambda}\left(\sqrt{\mu-\lambda}\ V_\mu\right) +$$

$$2V_\mu \sqrt{\frac{(a^2-\mu)(b^2-\mu)(c^2-\mu)}{(\lambda-\mu)(\nu-\mu)}}\ \frac{\partial V_\mu}{\partial \mu} +$$

$$\frac{2V_\nu}{\nu-\mu} \sqrt{\frac{(a^2-\nu)(b^2-\nu)(c^2-\nu)}{\nu-\lambda}}\ \frac{\partial}{\partial \nu}\left(\sqrt{\nu-\mu}\ V_\mu\right) -$$

$$\frac{V_\lambda^2}{\mu-\lambda} \sqrt{\frac{(a^2-\mu)(b^2-\mu)(c^2-\mu)}{(\lambda-\mu)(\nu-\mu)}} - \frac{V_\nu^2}{\mu-\nu} \sqrt{\frac{(a^2-\mu)(b^2-\mu)(c^2-\mu)}{(\lambda-\mu)(\nu-\mu)}}\ , \qquad (208)$$

$$\hat{f}_\nu = \frac{\partial V_\nu}{\partial t} + \frac{2V_\nu}{\lambda-\nu} \sqrt{\frac{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}{\mu-\lambda}}\ \frac{\partial}{\partial \lambda}\left(\sqrt{\nu-\lambda}\ V_\nu\right) +$$

$$\frac{2V_\mu}{\nu-\mu} \sqrt{\frac{(a^2-\mu)(b^2-\mu)(\mu-c^2)}{\mu-\lambda}}\ \frac{\partial}{\partial \mu}\left(\sqrt{\nu-\mu}\ V_\nu\right) +$$

$$2V_\nu \sqrt{\frac{(a^2-\nu)(b^2-\nu)(c^2-\nu)}{(\lambda-\nu)(\mu-\nu)}}\ \frac{\partial V_\nu}{\partial \nu} -$$

$$\frac{V_\lambda^2}{\nu-\lambda} \sqrt{\frac{(a^2-\nu)(b^2-\nu)(c^2-\nu)}{(\lambda-\nu)(\mu-\nu)}} - \frac{V_\mu^2}{\nu-\mu} \sqrt{\frac{(a^2-\nu)(b^2-\nu)(c^2-\nu)}{(\lambda-\nu)(\mu-\nu)}}\ , \qquad (209)$$

$$
\begin{aligned}
\frac{g^{jk}}{\sqrt{g_{11}}}\,\tau_{1j,k} = {} & -2\sqrt{\frac{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}{(\mu-\lambda)(\nu-\lambda)}}\;\frac{\partial p}{\partial \lambda} \; + \\
& \frac{2}{\sqrt{(\mu-\lambda)(\nu-\lambda)}}\,\frac{\partial}{\partial \lambda}\left(\sqrt{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}\;\hat{\tau}^{\lambda\lambda}\right) + \\
& \frac{2}{\nu-\mu}\sqrt{\frac{(a^2-\mu)(b^2-\mu)(\mu-c^2)}{\mu-\lambda}}\;\frac{\partial}{\partial \mu}\left(\sqrt{\nu-\mu}\;\hat{\tau}^{\lambda\mu}\right) + \\
& \frac{2}{\nu-\mu}\sqrt{\frac{(a^2-\nu)(b^2-\nu)(c^2-\nu)}{\nu-\lambda}}\;\frac{\partial}{\partial \nu}\left(\sqrt{\nu-\mu}\;\hat{\tau}^{\lambda\nu}\right) + \\
& \frac{(2\lambda-\mu-\nu)\sqrt{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}}{(\mu-\lambda)^{3/2}\,(\nu-\lambda)^{3/2}}\;\hat{\tau}^{\lambda\lambda} \; + \\
& \frac{(a^2-\lambda)(b^2-\lambda)+(a^2-\lambda)(c^2-\lambda)+(b^2-\lambda)(c^2-\lambda)}{\sqrt{(\mu-\lambda)(\nu-\lambda)(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}}\;\hat{\tau}^{\lambda\lambda} \; + \\
& \frac{\sqrt{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}}{\sqrt{\nu-\lambda}\;(\mu-\lambda)^{3/2}}\;\hat{\tau}^{\mu\mu} + \frac{\sqrt{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}}{\sqrt{\mu-\lambda}\;(\nu-\lambda)^{3/2}}\;\hat{\tau}^{\nu\nu} + \\
& \frac{2\sqrt{(a^2-\mu)(b^2-\mu)(\mu-c^2)}}{(\mu-\lambda)^{3/2}\sqrt{\nu-\mu}}\;\hat{\tau}^{\lambda\mu} + \frac{2\sqrt{(a^2-\nu)(b^2-\nu)(c^2-\nu)}}{(\nu-\lambda)^{3/2}\sqrt{\nu-\mu}}\;\hat{\tau}^{\lambda\nu} \, ,
\end{aligned}
\qquad (210)
$$

$$\frac{g^{jk}}{\sqrt{g_{22}}} \tau_{2j,k} = -2\sqrt{\frac{(a^2-\mu)(b^2-\mu)(c^2-\mu)}{(\lambda-\mu)(\nu-\mu)}}\frac{\partial p}{\partial \mu} +$$

$$\frac{2}{\nu-\lambda}\sqrt{\frac{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}{\mu-\lambda}}\frac{\partial}{\partial\lambda}\left(\sqrt{\nu-\lambda}\,\hat{\pi}^{\mu\lambda}\right) +$$

$$\frac{2}{\sqrt{(\mu-\lambda)(\nu-\mu)}}\frac{\partial}{\partial\mu}\left(\sqrt{(a^2-\mu)(b^2-\mu)(\mu-c^2)}\,\hat{\pi}^{\mu\mu}\right) +$$

$$\frac{2}{\nu-\lambda}\sqrt{\frac{(a^2-\nu)(b^2-\nu)(c^2-\nu)}{\nu-\mu}}\frac{\partial}{\partial\nu}\left(\sqrt{\nu-\lambda}\,\hat{\pi}^{\mu\nu}\right) -$$

$$\frac{\sqrt{(a^2-\mu)(b^2-\mu)(c^2-\mu)}}{\sqrt{\nu-\mu}\,(\mu-\lambda)^{3/2}}\hat{\pi}^{\lambda\lambda} - \frac{(2\mu-\lambda-\nu)\sqrt{(a^2-\mu)(b^2-\mu)(\mu-c^2)}}{(\nu-\mu)^{3/2}(\mu-\lambda)^{3/2}}\hat{\pi}^{\mu\mu} -$$

$$\frac{(a^2-\mu)(b^2-\mu)+(a^2-\mu)(c^2-\mu)+(b^2-\mu)(c^2-\mu)}{\sqrt{(\mu-\lambda)(\nu-\mu)(a^2-\mu)(b^2-\mu)(\mu-c^2)}}\hat{\pi}^{\mu\mu} +$$

$$\frac{\sqrt{(a^2-\mu)(b^2-\mu)(\mu-c^2)}}{\sqrt{\mu-\lambda}\,(\nu-\mu)^{3/2}}\hat{\pi}^{\nu\nu} + \frac{2\sqrt{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}}{\sqrt{\nu-\lambda}\,(\mu-\lambda)^{3/2}}\hat{\pi}^{\mu\lambda} +$$

$$\frac{2\sqrt{(a^2-\nu)(b^2-\nu)(c^2-\nu)}}{\sqrt{\nu-\lambda}\,(\nu-\mu)^{3/2}}\hat{\pi}^{\mu\nu}, \tag{211}$$

and

$$\frac{g^{jk}}{\sqrt{g_{33}}}\,\tau_{3j,k} = -2\sqrt{\frac{(a^2-\nu)(b^2-\nu)(c^2-\nu)}{(\lambda-\nu)(\mu-\nu)}}\;\frac{\partial p}{\partial \nu} +$$

$$\frac{2}{\mu-\lambda}\sqrt{\frac{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}{\nu-\lambda}}\;\frac{\partial}{\partial\lambda}\left(\sqrt{\mu-\lambda}\;\hat{\tau}^{\nu\lambda}\right) +$$

$$\frac{2}{\mu-\lambda}\sqrt{\frac{(a^2-\mu)(b^2-\mu)(c^2-\mu)}{\mu-\nu}}\;\frac{\partial}{\partial\mu}\left(\sqrt{\mu-\lambda}\;\hat{\tau}^{\nu\mu}\right) +$$

$$\frac{2}{\sqrt{(\lambda-\nu)(\mu-\nu)}}\;\frac{\partial}{\partial\nu}\left(\sqrt{(a^2-\nu)(b^2-\nu)(c^2-\nu)}\;\hat{\tau}^{\nu\nu}\right) -$$

$$\frac{\sqrt{(a^2-\nu)(b^2-\nu)(c^2-\nu)}}{(\nu-\lambda)^{3/2}\sqrt{\nu-\mu}}\;\hat{\tau}^{\lambda\lambda} - \frac{\sqrt{(a^2-\nu)(b^2-\nu)(c^2-\nu)}}{\sqrt{\nu-\lambda}\,(\nu-\mu)^{3/2}}\;\hat{\tau}^{\mu\mu} +$$

$$\frac{(2\nu-\lambda-\mu)\sqrt{(a^2-\nu)(b^2-\nu)(c^2-\nu)}}{(\nu-\lambda)^{3/2}(\nu-\mu)^{3/2}}\;\hat{\tau}^{\nu\nu} +$$

$$\frac{(a^2-\nu)(b^2-\nu)+(a^2-\nu)(c^2-\nu)+(b^2-\nu)(c^2-\nu)}{\sqrt{(a^2-\nu)(b^2-\nu)(c^2-\nu)(\lambda-\nu)(\mu-\nu)}}\;\hat{\tau}^{\nu\nu} +$$

$$\frac{2\sqrt{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}}{\sqrt{\mu-\lambda}\,(\nu-\lambda)^{3/2}}\;\hat{\tau}^{\nu\lambda} + \frac{2\sqrt{(a^2-\mu)(b^2-\mu)(\mu-c^2)}}{\sqrt{\mu-\lambda}\,(\nu-\mu)^{3/2}}\;\hat{\tau}^{\nu\mu}. \qquad (212)$$

Confocal Parabolic Coordinates

$$x^1 = \lambda \qquad x^2 = \mu \qquad x^3 = \nu \qquad (213)$$

$$g_{11} = \frac{(\lambda-\mu)(\lambda-\nu)}{4(a-\lambda)(b-\lambda)} \qquad (214)$$

$$g_{22} = \frac{(\mu-\lambda)(\mu-\nu)}{4(a-\mu)(b-\mu)} \qquad (215)$$

$$g_{33} = \frac{(\nu-\lambda)(\nu-\mu)}{4(a-\nu)(b-\nu)} \tag{216}$$

$$g = \frac{(\lambda-\mu)^2(\mu-\nu)^2(\lambda-\nu)^2}{64(a-\lambda)(b-\lambda)(\mu-a)(b-\mu)(a-\nu)(b-\nu)} \tag{217}$$

Continuity Equation.

$$\frac{\partial \rho}{\partial t} + \frac{2\sqrt{(a-\lambda)(b-\lambda)}}{(\mu-\lambda)(\nu-\lambda)} \frac{\partial}{\partial \lambda}\left(\sqrt{(\mu-\lambda)(\nu-\lambda)}\ \rho V_\lambda\right) +$$

$$\frac{2\sqrt{(\mu-a)(b-\mu)}}{(\lambda-\mu)(\mu-\nu} \frac{\partial}{\partial \mu}\left(\sqrt{(\lambda-\mu)(\mu-\nu)}\ \rho V_\mu\right) +$$

$$\frac{2\sqrt{(a-\nu)(b-\nu)}}{(\lambda-\nu)(\mu-\nu)} \frac{\partial}{\partial \nu}\left(\sqrt{(\lambda-\nu)(\mu-\nu)}\ \rho V_\nu\right) = 0 \tag{218}$$

Energy Equation. The left side of equation (5) becomes

$$\rho\left\{\frac{\partial}{\partial t}\left[h + \frac{1}{2}(V_\lambda^2 + V_\mu^2 + V_\nu^2)\right] + 2V_\lambda\sqrt{\frac{(a-\lambda)(b-\lambda)}{(\lambda-\mu)(\lambda-\nu)}}\frac{\partial}{\partial \lambda}\left[h + \right.\right.$$

$$\left.\frac{1}{2}(V_\lambda^2 + V_\mu^2 + V_\nu^2)\right] + 2V_\mu\sqrt{\frac{(a-\mu)(b-\mu)}{(\mu-\lambda)(\mu-\nu)}}\frac{\partial}{\partial \mu}\left[h + \right.$$

$$\left.\frac{1}{2}(V_\lambda^2 + V_\mu^2 + V_\nu^2)\right] + 2V_\nu\sqrt{\frac{(a-\nu)(b-\nu)}{(\nu-\lambda)(\nu-\mu)}}\frac{\partial}{\partial \nu}\left[h + \right.$$

$$\left.\left.\frac{1}{2}(V_\lambda^2 + V_\mu^2 + V_\nu^2)\right]\right\}. \tag{219}$$

Also,

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(\frac{\sqrt{g}}{\sqrt{g_{(j)(j)}}}V_i\hat{\pi}^{ij}\right) = \frac{2\sqrt{(a-\lambda)(b-\lambda)}}{(\mu-\lambda)(\nu-\lambda)}\frac{\partial}{\partial\lambda}\Big[\sqrt{(\mu-\lambda)(\nu-\lambda)}\,(V_\lambda\hat{\pi}^{\lambda\lambda} +$$

$$V_\mu\hat{\pi}^{\mu\lambda} + V_\nu\hat{\pi}^{\nu\lambda})\Big] + \frac{2\sqrt{(\mu-a)(b-\mu)}}{(\lambda-\mu)(\mu-\nu)}\frac{\partial}{\partial\mu}\Big[\sqrt{(\lambda-\mu)(\mu-\nu)}\,(V_\lambda\hat{\pi}^{\lambda\mu} +$$

$$V_\mu\hat{\pi}^{\mu\mu} + V_\nu\hat{\pi}^{\nu\mu})\Big] + \frac{2\sqrt{(a-\nu)(b-\nu)}}{(\lambda-\nu)(\mu-\nu)}\frac{\partial}{\partial\nu}\Big[\sqrt{(\lambda-\nu)(\mu-\nu)}\,(V_\lambda\hat{\pi}^{\lambda\nu} +$$

$$V_\mu\hat{\pi}^{\mu\nu} + V_\nu\hat{\pi}^{\nu\nu})\Big], \qquad (220)$$

$$\rho V_i\hat{F}_i = \rho\left[V_\lambda\hat{F}_\lambda + V_\mu\hat{F}_\mu + V_\nu\hat{F}_\nu\right], \qquad (221)$$

and

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\frac{\sqrt{g}}{\sqrt{g_{(i)(i)}}}\hat{g}_i\right) = \frac{2\sqrt{(a-\lambda)(b-\lambda)}}{(\mu-\lambda)(\nu-\lambda)}\frac{\partial}{\partial\lambda}\left(\sqrt{(\mu-\lambda)(\nu-\lambda)}\,\hat{g}_\lambda\right) +$$

$$\frac{2\sqrt{(\mu-a)(b-\mu)}}{(\lambda-\mu)(\mu-\nu)}\frac{\partial}{\partial\mu}\left(\sqrt{(\lambda-\mu)(\mu-\nu)}\,\hat{g}_\mu\right) +$$

$$\frac{2\sqrt{(a-\nu)(b-\nu)}}{(\lambda-\nu)(\mu-\nu)}\frac{\partial}{\partial\nu}\left(\sqrt{(\lambda-\nu)(\mu-\nu)}\,\hat{g}_\nu\right). \qquad (222)$$

The components of the stress tensor are

$$\hat{\pi}^{\lambda\mu} = \hat{\pi}^{\mu\lambda} = \tau\Big[2\sqrt{\frac{(a-\lambda)(b-\lambda)}{\lambda-\nu}}\frac{\partial}{\partial\lambda}\left(\frac{1}{\sqrt{\lambda-\mu}}V_\mu\right) +$$

$$2\sqrt{\frac{(\mu-a)(b-\mu)}{\mu-\nu}}\frac{\partial}{\partial\mu}\left(\frac{1}{\sqrt{\lambda-\mu}}V_\lambda\right)\Big], \qquad (223)$$

$$\hat{\Pi}^{\lambda\nu} = \hat{\Pi}^{\nu\lambda} = \tau\Big[2\sqrt{\frac{(a-\lambda)(b-\lambda)}{\lambda-\mu}}\,\frac{\partial}{\partial\lambda}\Big(\frac{1}{\sqrt{\lambda-\nu}}\,V_\nu\Big) +$$

$$2\sqrt{\frac{(a-\nu)(b-\nu)}{\mu-\nu}}\,\frac{\partial}{\partial\nu}\Big(\frac{1}{\sqrt{\lambda-\nu}}\,V_\lambda\Big)\Big], \qquad (224)$$

$$\hat{\Pi}^{\mu\nu} = \hat{\Pi}^{\nu\mu} = \tau\Big[2\sqrt{\frac{(\mu-a)(b-\mu)}{\lambda-\mu}}\,\frac{\partial}{\partial\mu}\Big(\frac{1}{\sqrt{\mu-\nu}}\,V_\nu\Big) +$$

$$2\sqrt{\frac{(a-\nu)(b-\nu)}{\lambda-\nu}}\,\frac{\partial}{\partial\mu}\Big(\frac{1}{\sqrt{\mu-\nu}}\,V_\mu\Big)\Big], \qquad (225)$$

$$\hat{\Pi}^{\lambda\lambda} = \Big(\eta - \frac{2}{3}\tau\Big)\Big[\frac{2\sqrt{(a-\lambda)(b-\lambda)}}{(\lambda-\mu)(\lambda-\nu)}\,\frac{\partial}{\partial\lambda}\Big(\sqrt{(\mu-\lambda)(\nu-\lambda)}\,V_\lambda\Big) +$$

$$\frac{2\sqrt{(\mu-a)(b-\mu)}}{(\lambda-\mu)(\nu-\mu)}\,\frac{\partial}{\partial\mu}\Big(\sqrt{(\lambda-\mu)(\mu-\nu)}\,V_\mu\Big) +$$

$$\frac{2\sqrt{(a-\nu)(b-\nu)}}{(\lambda-\nu)(\mu-\nu)}\,\frac{\partial}{\partial\nu}\Big(\sqrt{(\lambda-\nu)(\mu-\nu)}\,V_\nu\Big)\Big] +$$

$$\tau\Big[4\frac{\partial}{\partial\lambda}\Big(\sqrt{\frac{(a-\lambda)(b-\lambda)}{(\lambda-\mu)(\lambda-\nu)}}\,V_\lambda + \frac{2\sqrt{(a-\lambda)(b-\lambda)}\,(2\lambda-\mu-\nu)}{(\lambda-\mu)^{3/2}(\lambda-\nu)^{3/2}}\,V_\lambda -$$

$$\frac{2(2\lambda-a-b)}{\sqrt{(a-\lambda)(b-\lambda)(\lambda-\mu)(\lambda-\nu)}}\,V_\lambda - \frac{2}{\lambda-\mu}\sqrt{\frac{(a-\mu)(b-\mu)}{(\mu-\lambda)(\mu-\nu)}}\,V_\mu -$$

$$\frac{2}{\lambda-\nu}\sqrt{\frac{(a-\nu)(b-\nu)}{(\nu-\lambda)(\nu-\mu)}}\,V_\nu\Big], \qquad (226)$$

$$\hat{\Pi}^{\mu\mu} = (\eta - \tfrac{2}{3}\tau)\Big[\frac{2\sqrt{(a-\lambda)(b-\lambda)}}{(\lambda-\mu)(\lambda-\nu)}\frac{\partial}{\partial\lambda}\Big(\sqrt{(\lambda-\mu)(\lambda-\nu)}\;V_\lambda\Big) +$$

$$\frac{2\sqrt{(\mu-a)(b-\mu)}}{(\lambda-\mu)(\mu-\nu)}\frac{\partial}{\partial\mu}\Big(\sqrt{(\lambda-\mu)(\mu-\nu)}\;V_\mu\Big) +$$

$$\frac{2\sqrt{(a-\nu)(b-\nu)}}{(\mu-\nu)(\lambda-\nu)}\frac{\partial}{\partial\nu}\Big(\sqrt{(\lambda-\nu)(\mu-\nu)}\;V_\nu\Big) +$$

$$\tau\Big[4\frac{\partial}{\partial\mu}\Big(\sqrt{\frac{(a-\mu)(b-\mu)}{(\mu-\lambda)(\mu-\nu)}}\;V_\mu\Big) - \frac{2}{\mu-\lambda}\sqrt{\frac{(a-\lambda)(b-\lambda)}{(\lambda-\mu)(\lambda-\nu)}}\;V_\lambda +$$

$$\frac{2\sqrt{(\mu-a)(b-\mu)}\;(2\mu-\lambda-\nu)}{(\lambda-\mu)^{3/2}\;(\mu-\nu)^{3/2}}\;V_\mu -$$

$$\frac{2\,(2\mu-a-b)}{\sqrt{(a-\mu)(b-\mu)(\mu-\lambda)(\mu-\nu)}}\;V_\mu -$$

$$\frac{2}{\mu-\nu}\sqrt{\frac{(a-\nu)(b-\nu)}{(\nu-\lambda)(\nu-\mu)}}\;V_\nu\Big]\;, \qquad (227)$$

and

$$
\begin{aligned}
\hat{\Pi}^{\nu\nu} = {} & (\eta - \tfrac{2}{3}\sigma)\Big[\frac{2\sqrt{(a-\lambda)(b-\lambda)}}{(\lambda-\nu)(\lambda-\mu)}\frac{\partial}{\partial\lambda}\left(\sqrt{(\lambda-\mu)(\lambda-\nu)}\,V_\lambda\right) + \\
& \frac{2\sqrt{(\mu-a)(b-\mu)}}{(\mu-\nu)(\lambda-\mu)}\frac{\partial}{\partial\mu}\left(\sqrt{(\lambda-\mu)(\mu-\nu)}\,V_\mu\right) + \\
& \frac{2\sqrt{(a-\nu)(b-\nu)}}{(\nu-\lambda)(\nu-\mu)}\frac{\partial}{\partial\nu}\left(\sqrt{(\lambda-\nu)(\mu-\nu)}\,V_\nu\right)\Big] + \\
& \sigma\Big[4\frac{\partial}{\partial\nu}\left(\sqrt{\frac{(a-\nu)(b-\nu)}{(\nu-\lambda)(\nu-\mu)}}\,V_\nu\right) - \frac{2}{\nu-\lambda}\sqrt{\frac{(a-\lambda)(b-\lambda)}{(\lambda-\mu)(\lambda-\nu)}}\,V_\lambda - \\
& \frac{2}{\nu-\mu}\sqrt{\frac{(a-\mu)(b-\mu)}{(\mu-\lambda)(\mu-\nu)}}\,V_\mu + \frac{2\sqrt{(a-\nu)(b-\nu)}\,(2\nu-\lambda-\mu)}{(\lambda-\nu)^{3/2}(\mu-\nu)^{3/2}}\,V_\nu - \\
& \frac{2(2\nu-a-b)}{\sqrt{(a-\nu)(b-\nu)(\nu-\lambda)(\nu-\mu)}}\,V_\nu\Big] \;. \qquad (228)
\end{aligned}
$$

Equation of Motion. The terms of interest in the equation of motion are

$$
\begin{aligned}
\hat{f}_\lambda = {} & \frac{\partial V_\lambda}{\partial t} + 2V_\lambda\sqrt{\frac{(a-\lambda)(b-\lambda)}{(\lambda-\mu)(\lambda-\nu)}}\,\frac{\partial V_\lambda}{\partial\lambda} + \\
& \frac{2V_\mu}{\lambda-\mu}\sqrt{\frac{(\mu-a)(b-\mu)}{\mu-\nu}}\,\frac{\partial}{\partial\mu}\left(\sqrt{\lambda-\mu}\,V_\lambda\right) + \\
& \frac{2V_\nu}{\lambda-\nu}\sqrt{\frac{(a-\nu)(b-\nu)}{\mu-\nu}}\,\frac{\partial}{\partial\nu}\left(\sqrt{\lambda-\nu}\,V_\lambda\right) + \\
& \frac{V_\mu^2}{\mu-\lambda}\sqrt{\frac{(a-\lambda)(b-\lambda)}{(\lambda-\mu)(\lambda-\nu)}} + \frac{V_\nu^2}{\nu-\lambda}\sqrt{\frac{(a-\lambda)(b-\lambda)}{(\lambda-\mu)(\lambda-\nu)}} \;, \qquad (229)
\end{aligned}
$$

$$\hat{f}_{\mu} = \frac{\partial V_\mu}{\partial t} + \frac{2V_\lambda}{\mu-\lambda}\sqrt{\frac{(a-\lambda)(b-\lambda)}{\lambda-\nu}}\frac{\partial}{\partial\lambda}\left(\sqrt{\lambda-\mu}\,V_\mu\right) +$$

$$2V_\mu\sqrt{\frac{(a-\mu)(b-\mu)}{(\mu-\lambda)(\mu-\nu)}}\frac{\partial V_\mu}{\partial\mu} +$$

$$\frac{2V_\nu}{\mu-\nu}\sqrt{\frac{(a-\nu)(b-\nu)}{\lambda-\nu}}\frac{\partial}{\partial\nu}\left(\sqrt{\mu-\nu}\,V_\mu\right) +$$

$$\frac{V_\lambda^2}{\lambda-\mu}\sqrt{\frac{(a-\mu)(b-\mu)}{(\mu-\lambda)(\mu-\nu)}} + \frac{V_\nu^2}{\nu-\mu}\sqrt{\frac{(a-\mu)(b-\mu)}{(\mu-\lambda)(\mu-\nu)}} \quad , \qquad (230)$$

$$\hat{f}_{\nu} = \frac{\partial V_\nu}{\partial t} + \frac{2V_\lambda}{\nu-\lambda}\sqrt{\frac{(a-\lambda)(b-\lambda)}{\lambda-\mu}}\frac{\partial}{\partial\lambda}\left(\sqrt{\lambda-\nu}\,V_\nu\right) +$$

$$\frac{2V_\mu}{\mu-\nu}\sqrt{\frac{(\mu-a)(b-\mu)}{\lambda-\mu}}\frac{\partial}{\partial\mu}\left(\sqrt{\mu-\nu}\,V_\nu\right) +$$

$$2V_\nu\sqrt{\frac{(a-\nu)(b-\nu)}{(\nu-\lambda)(\nu-\mu)}}\frac{\partial V_\nu}{\partial\nu} + \frac{V_\lambda^2}{\lambda-\nu}\sqrt{\frac{(a-\nu)(b-\nu)}{(\nu-\lambda)(\nu-\mu)}} +$$

$$\frac{V_\mu^2}{\mu-\nu}\sqrt{\frac{(a-\nu)(b-\nu)}{(\nu-\lambda)(\nu-\mu)}} \quad , \qquad (231)$$

$$
\begin{aligned}
\frac{g^{jk}}{\sqrt{g_{11}}}\,\tau_{1j,k} = {} & -2\sqrt{\frac{(a-\lambda)(b-\lambda)}{(\lambda-\mu)(\lambda-\nu)}}\,\frac{\partial p}{\partial \lambda} + \\
& \frac{2}{\sqrt{(\lambda-\mu)(\lambda-\nu)}}\,\frac{\partial}{\partial\lambda}\left(\sqrt{(a-\lambda)(b-\lambda)}\;\hat{\pi}^{\lambda\lambda}\right) + \\
& \frac{2}{\nu-\mu}\sqrt{\frac{(\mu-a)(b-\mu)}{\lambda-\mu}}\,\frac{\partial}{\partial\mu}\left(\sqrt{\mu-\nu}\;\hat{\pi}^{\lambda\mu}\right) + \\
& \frac{2}{\nu-\mu}\sqrt{\frac{(a-\nu)(b-\nu)}{\lambda-\nu}}\,\frac{\partial}{\partial\nu}\left(\sqrt{\mu-\nu}\;\hat{\pi}^{\lambda\nu}\right) + \\
& \frac{(2\lambda-\mu-\nu)\sqrt{(a-\lambda)(b-\lambda)}}{(\lambda-\mu)^{3/2}(\lambda-\mu)^{3/2}}\,\hat{\pi}^{\lambda\lambda} + \\
& \frac{a+b-2\lambda}{\sqrt{(a-\lambda)(b-\lambda)(\lambda-\mu)(\lambda-\nu)}}\,\hat{\pi}^{\lambda\lambda} + \frac{\sqrt{(\lambda-a)(\lambda-b)}}{(\lambda-\mu)^{3/2}\sqrt{\lambda-\nu}}\,\hat{\pi}^{\mu\mu} + \\
& \frac{\sqrt{(a-\lambda)(b-\lambda)}}{(\lambda-\nu)^{3/2}\sqrt{\lambda-\mu}}\,\hat{\pi}^{\nu\nu} - \frac{2}{(\lambda-\mu)^{3/2}}\sqrt{\frac{(a-\mu)(b-\mu)}{\nu-\mu}}\,\hat{\pi}^{\lambda\mu} - \\
& \frac{2}{(\lambda-\nu)^{3/2}}\sqrt{\frac{(a-\nu)(b-\nu)}{\mu-\nu}}\,\hat{\pi}^{\lambda\nu}\,, \qquad (232)
\end{aligned}
$$

$$\frac{g^{jk}}{\sqrt{g_{22}}} \tau_{2j,k} = -2\sqrt{\frac{(a-\mu)(b-\mu)}{(\mu-\lambda)(\mu-\nu)}}\frac{\partial p}{\partial \mu} +$$

$$\frac{2}{\sqrt{(\lambda-\mu)(\mu-\nu)}}\frac{\partial}{\partial \mu}\left(\sqrt{(\mu-a)(b-\mu)}\,\hat{\Pi}^{\mu\mu}\right) +$$

$$\frac{2}{\lambda-\nu}\sqrt{\frac{(a-\lambda)(b-\lambda)}{\lambda-\mu}}\frac{\partial}{\partial \lambda}\left(\sqrt{\lambda-\nu}\,\hat{\Pi}^{\mu\lambda}\right) +$$

$$\frac{2}{\nu-\lambda}\sqrt{\frac{(a-\nu)(b-\nu)}{\mu-\nu}}\frac{\partial}{\partial \nu}\left(\sqrt{\lambda-\nu}\,\hat{\Pi}^{\mu\nu}\right) +$$

$$\frac{\sqrt{(\mu-a)(b-\mu)}}{(\lambda-\mu)^{3/2}\sqrt{\mu-\nu}}\hat{\Pi}^{\lambda\lambda} - \frac{(2\mu-\lambda-\nu)\sqrt{(\mu-a)(b-\mu)}}{(\lambda-\mu)^{3/2}(\mu-\nu)^{3/2}}\hat{\Pi}^{\mu\mu} +$$

$$\frac{a+b-2\mu}{\sqrt{(a-\mu)(b-\mu)(\mu-\lambda)(\mu-\nu)}}\hat{\Pi}^{\mu\mu} -$$

$$\frac{\sqrt{(\mu-a)(b-\mu)}}{(\mu-\nu)^{3/2}\sqrt{\lambda-\mu}}\hat{\Pi}^{\nu\nu} - \frac{2}{(\lambda-\mu)^{3/2}}\sqrt{\frac{(a-\lambda)(b-\lambda)}{\lambda-\nu}}\hat{\Pi}^{\mu\lambda} -$$

$$\frac{2}{(\mu-\nu)^{3/2}}\sqrt{\frac{(a-\nu)(b-\nu)}{\lambda-\nu}}\hat{\Pi}^{\mu\nu}, \qquad (233)$$

and

$$\frac{g^{jk}}{\sqrt{g_{33}}}\tau_{3j,k} = -2\sqrt{\frac{(a-\nu)(b-\nu)}{(\nu-\lambda)(\nu-\mu)}}\frac{\partial p}{\partial \nu} +$$

$$\frac{2}{\lambda-\mu}\sqrt{\frac{(a-\lambda)(b-\lambda)}{\lambda-\nu}}\frac{\partial}{\partial\lambda}\left(\sqrt{\lambda-\mu}\;\hat{\pi}^{\nu\lambda}\right) +$$

$$\frac{2}{\mu-\lambda}\sqrt{\frac{(a-\mu)(b-\mu)}{\nu-\mu}}\frac{\partial}{\partial\mu}\left(\sqrt{\lambda-\mu}\;\hat{\pi}^{\nu\mu}\right) +$$

$$\frac{2}{\sqrt{(\nu-\lambda)(\nu-\mu)}}\frac{\partial}{\partial\nu}\left(\sqrt{(a-\nu)(b-\nu)}\;\hat{\pi}^{\nu\nu}\right) +$$

$$\frac{\sqrt{(a-\nu)(b-\nu)}}{(\lambda-\nu)^{3/2}\sqrt{\mu-\nu}}\hat{\pi}^{\lambda\lambda} + \frac{\sqrt{(a-\nu)(b-\nu)}}{(\mu-\nu)^{3/2}\sqrt{\lambda-\nu}}\hat{\pi}^{\mu\mu} +$$

$$\frac{(2\nu-\lambda-\mu)\sqrt{(a-\nu)(b-\nu)}}{(\lambda-\nu)^{3/2}(\mu-\nu)^{3/2}}\hat{\pi}^{\nu\nu} +$$

$$\frac{a+b-2\nu}{\sqrt{(a-\nu)(b-\nu)(\nu-\lambda)(\nu-\mu)}}\hat{\pi}^{\nu\nu} -$$

$$\frac{2}{(\lambda-\nu)^{3/2}}\sqrt{\frac{(a-\lambda)(b-\lambda)}{\lambda-\mu}}\hat{\pi}^{\nu\lambda} -$$

$$\frac{2}{(\mu-\nu)^{3/2}}\sqrt{\frac{(a-\mu)(b-\mu)}{\mu-\lambda}}\hat{\pi}^{\nu\mu}. \qquad (234)$$

Conical Coordinates

$$x^1 = \lambda \qquad x^2 = \mu \qquad x^3 = \nu \qquad (235)$$

$$g_{11} = 1 \qquad (236)$$

$$g_{22} = \frac{\lambda^2(\mu^2-\nu^2)}{(\mu^2-b^2)(c^2-\mu^2)} \tag{237}$$

$$g_{33} = \frac{\lambda^2(\mu^2-\nu^2)}{(b^2-\nu^2)(c^2-\nu^2)} \tag{238}$$

$$g = \frac{\lambda^4(\mu^2-\nu^2)^2}{(\mu^2-b^2)(c^2-\mu^2)(b^2-\nu^2)(c^2-\nu^2)} \tag{239}$$

Continuity Equation.

$$\frac{\partial \rho}{\partial t} + \frac{1}{\lambda^2}\frac{\partial}{\partial \lambda}\left(\lambda^2 \rho V_\lambda\right) +$$

$$\frac{1}{\lambda}\frac{\sqrt{(\mu^2-b^2)(c^2-\mu^2)}}{\mu^2-\nu^2}\frac{\partial}{\partial \mu}\left(\sqrt{\mu^2-\nu^2}\,\rho V_\mu\right) +$$

$$\frac{1}{\lambda}\frac{\sqrt{(b^2-\nu^2)(c^2-\nu^2)}}{\mu^2-\nu^2}\frac{\partial}{\partial \nu}\left(\sqrt{\mu^2-\nu^2}\,\rho V_\nu\right) = 0 \tag{240}$$

Energy Equation. The left side of equation (5) becomes

$$\rho\left\{\frac{\partial}{\partial t}\left[h+\frac{1}{2}\left(V_\lambda^2+V_\mu^2+V_\nu^2\right)\right]+V_\lambda\frac{\partial}{\partial\lambda}\left[h+\right.\right.$$
$$\left.\frac{1}{2}\left(V_\lambda^2+V_\mu^2+V_\nu^2\right)\right]+\frac{V_\mu}{\lambda}\sqrt{\frac{(\mu^2-b^2)(c^2-\mu^2)}{\mu^2-\nu^2}}\frac{\partial}{\partial\mu}\left[h+\right.$$
$$\left.\frac{1}{2}\left(V_\lambda^2+V_\mu^2+V_\nu^2\right)\right]+\frac{V_\nu}{\lambda}\sqrt{\frac{(b^2-\nu^2)(c^2-\nu^2)}{\mu^2-\nu^2}}\frac{\partial}{\partial\nu}\left[h+\right.$$
$$\left.\left.\frac{1}{2}\left(V_\lambda^2+V_\mu^2+V_\nu^2\right)\right]\right\} . \tag{241}$$

Also,

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(\frac{\sqrt{g}}{\sqrt{g_{(j)(j)}}}V_i\hat{\pi}^{ij}\right)=\frac{1}{\lambda^2}\frac{\partial}{\partial\lambda}\left[\lambda^2\left(V_\lambda\hat{\pi}^{\lambda\lambda}+V_\mu\hat{\pi}^{\mu\lambda}+\right.\right.$$
$$\left.\left.V_\nu\hat{\pi}^{\nu\lambda}\right)\right]+\frac{\sqrt{(\mu^2-b^2)(c^2-\mu^2)}}{\lambda(\mu^2-\nu^2)}\frac{\partial}{\partial\mu}\left[\sqrt{\mu^2-\nu^2}\left(V_\lambda\hat{\pi}^{\lambda\mu}+V_\mu\hat{\pi}^{\mu\mu}+\right.\right.$$
$$\left.\left.V_\nu\hat{\pi}^{\nu\mu}\right)\right]+\frac{\sqrt{(b^2-\nu^2)(c^2-\nu^2)}}{\lambda(\mu^2-\nu^2)}\frac{\partial}{\partial\nu}\left[\sqrt{\mu^2-\nu^2}\left(V_\lambda\hat{\pi}^{\lambda\nu}+\right.\right.$$
$$\left.\left.V_\mu\hat{\pi}^{\mu\nu}+V_\nu\hat{\pi}^{\nu\nu}\right)\right] , \tag{242}$$

$$\rho V_i\hat{F}_i=\rho\left[V_\lambda\hat{F}_\lambda+V_\mu\hat{F}_\mu+V_\nu\hat{F}_\nu\right] , \tag{243}$$

and

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\frac{1}{\sqrt{g_{(i)(i)}}} \sqrt{g}\, \hat{q}_i\right) = \frac{1}{\lambda^2} \frac{\partial}{\partial \lambda}\left(\lambda^2 \hat{q}_\lambda\right) +$$

$$\frac{1}{\lambda} \frac{\sqrt{(\mu^2-b^2)(c^2-\mu^2)}}{\mu^2-\nu^2} \frac{\partial}{\partial \mu}\left(\sqrt{\mu^2-\nu^2}\, \hat{q}_\mu\right) +$$

$$\frac{1}{\lambda} \frac{\sqrt{(b^2-\nu^2)(c^2-\nu^2)}}{\mu^2-\nu^2} \frac{\partial}{\partial \nu}\left(\sqrt{\mu^2-\nu^2}\, \hat{q}_\nu\right). \qquad (244)$$

The components of the stress tensor are

$$\hat{\pi}^{\lambda\mu} = \pi^{\mu\lambda} = \tau\left[\lambda \frac{\partial}{\partial \lambda}\left(\frac{1}{\lambda} V_\mu\right) + \frac{1}{\lambda} \sqrt{\frac{(\mu^2-b^2)(c^2-\mu^2)}{\mu^2-\nu^2}} \frac{\partial V_\lambda}{\partial \mu}\right], \qquad (245)$$

$$\hat{\pi}^{\lambda\nu} = \hat{\pi}^{\nu\lambda} = \tau\left[\lambda \frac{\partial}{\partial \lambda}\left(\frac{1}{\lambda} V_\nu\right) + \frac{1}{\lambda} \sqrt{\frac{(b^2-\nu^2)(c^2-\nu^2)}{\mu^2-\nu^2}} \frac{\partial V_\lambda}{\partial \nu}\right], \qquad (246)$$

$$\hat{\pi}^{\mu\nu} = \hat{\pi}^{\nu\mu} = \tau\left[\frac{1}{\lambda} \sqrt{(\mu^2-b^2)(c^2-\mu^2)} \frac{\partial}{\partial \mu}\left(\frac{1}{\sqrt{\mu^2-\nu^2}} V_\nu\right) + \frac{1}{\lambda} \sqrt{(b^2-\nu^2)(c^2-\nu^2)} \frac{\partial}{\partial \nu}\left(\frac{1}{\sqrt{\mu^2-\nu^2}} V_\mu\right)\right], \qquad (247)$$

$$\hat{\Pi}^{\lambda\lambda} = \left(\eta - \tfrac{2}{3}\tau\right)\Big[\frac{1}{\lambda^2}\frac{\partial}{\partial\lambda}\left(\lambda^2 V_\lambda\right) +$$

$$\frac{\sqrt{(\mu^2-b^2)(c^2-\mu^2)}}{\lambda(\mu^2-\nu^2)}\frac{\partial}{\partial\mu}\left(\sqrt{\mu^2-\nu^2}\; V_\mu\right) +$$

$$\frac{\sqrt{(b^2-\nu^2)(c^2-\nu^2)}}{\lambda(\mu^2-\nu^2)}\frac{\partial}{\partial\nu}\left(\sqrt{\mu^2-\nu^2}\; V_\nu\right)\Big] + 2\tau\frac{\partial V_\lambda}{\partial\lambda}\,, \qquad (248)$$

$$\hat{\Pi}^{\mu\mu} = \left(\eta - \tfrac{2}{3}\tau\right)\Big[\frac{1}{\lambda^2}\frac{\partial}{\partial\lambda}\left(\lambda^2 V_\lambda\right) + \frac{\sqrt{(\mu^2-b^2)(c^2-\mu^2)}}{\lambda(\mu^2-\nu^2)}\frac{\partial}{\partial\mu}\left(\sqrt{\mu^2-\nu^2}\; V_\mu\right) +$$

$$\frac{\sqrt{(b^2-\nu^2)(c^2-\nu^2)}}{\lambda(\mu^2-\nu^2)}\frac{\partial}{\partial\nu}\left(\sqrt{\mu^2-\nu^2}\; V_\nu\right)\Big] + \tau\Big[\frac{2}{\lambda}\frac{\partial}{\partial\mu}\left(\sqrt{\frac{(\mu^2-b^2)(c^2-\mu^2)}{\mu^2-\nu^2}}\; V_\mu\right) +$$

$$\frac{2}{\lambda}V_\lambda + \frac{2\mu}{\lambda}\frac{\sqrt{(\mu^2-b^2)(c^2-\mu^2)}}{(\mu^2-\nu^2)^{3/2}}\,V_\mu -$$

$$\frac{2\mu}{\lambda}\frac{b^2+c^2-2\mu^2}{\sqrt{(\mu^2-b^2)(c^2-\mu^2)(\mu^2-\nu^2)}}\,V_\mu -$$

$$\frac{2\nu}{\lambda}\frac{\sqrt{(b^2-\nu^2)(c^2-\nu^2)}}{(\mu^2-\nu^2)^{3/2}}\,V_\nu\Big]\,, \qquad (249)$$

and

$$\hat{\pi}^{\nu\nu} = \left(\eta - \tfrac{2}{3}\tau\right)\left[\frac{1}{\lambda^2}\frac{\partial}{\partial\lambda}\left(\lambda^2 V_\lambda\right) + \right.$$

$$\frac{\sqrt{(\mu^2-b^2)(c^2-\mu^2)}}{\lambda(\mu^2-\nu^2)}\frac{\partial}{\partial\mu}\left(\sqrt{\mu^2-\nu^2}\,V_\mu\right) +$$

$$\left.\frac{\sqrt{(b^2-\nu^2)(c^2-\nu^2)}}{\lambda(\mu^2-\nu^2)}\frac{\partial}{\partial\nu}\left(\sqrt{\mu^2-\nu^2}\,V_\nu\right)\right] +$$

$$\tau\left[\frac{2}{\lambda}\frac{\partial}{\partial\nu}\left(\sqrt{\frac{(b^2-\nu^2)(c^2-\nu^2)}{\mu^2-\nu^2}}\,V_\nu\right) + \frac{2}{\lambda}V_\lambda + \right.$$

$$\frac{2\mu}{\lambda}\frac{\sqrt{(\mu^2-b^2)(c^2-\mu^2)}}{(\mu^2-\nu^2)^{3/2}}V_\mu - \frac{2\nu}{\lambda}\frac{\sqrt{(b^2-\nu^2)(c^2-\nu^2)}}{(\mu^2-\nu^2)^{3/2}}V_\nu +$$

$$\left.\frac{2\nu}{\lambda}\frac{b^2+c^2-2\nu^2}{\sqrt{(b^2-\nu^2)(c^2-\nu^2)(\mu^2-\nu^2)}}V_\nu\right]. \tag{250}$$

<u>Equation of Motion</u>. The terms of interest in the equation of motion are

$$\hat{f}_\lambda = \frac{\partial V_\lambda}{\partial t} + V_\lambda\frac{\partial V_\lambda}{\partial\lambda} + \frac{V_\mu}{\lambda}\sqrt{\frac{(\mu^2-b^2)(c^2-\mu^2)}{\mu^2-\nu^2}}\frac{\partial V_\lambda}{\partial\mu} +$$

$$\frac{V_\nu}{\lambda}\sqrt{\frac{(b^2-\nu^2)(c^2-\nu^2)}{\mu^2-\nu^2}}\frac{\partial V_\lambda}{\partial\nu} - \frac{V_\mu^2}{\lambda} - \frac{V_\nu^2}{\lambda}, \tag{251}$$

$$\hat{f}_\mu = \frac{\partial V_\mu}{\partial t} + \frac{V_\lambda}{\lambda}\frac{\partial}{\partial\lambda}(\lambda V_\mu) + \frac{V_\mu}{\lambda}\sqrt{\frac{(\mu^2-b^2)(c^2-\mu^2)}{\mu^2-\nu^2}}\frac{\partial V_\mu}{\partial\mu} +$$

$$\frac{V_\nu}{\lambda}\frac{\sqrt{(b^2-\nu^2)(c^2-\nu^2)}}{\mu^2-\nu^2}\frac{\partial}{\partial\nu}\left(\sqrt{\mu^2-\nu^2}\,V_\mu\right) -$$

$$\frac{V_\nu^{\,2}\mu}{\lambda(\mu^2-\nu^2)}\sqrt{\frac{(\mu^2-b^2)(c^2-\mu^2)}{\mu^2-\nu^2}}\,, \qquad (252)$$

$$\hat{f}_\nu = \frac{\partial V_\nu}{\partial t} + \frac{V_\lambda}{\lambda}\frac{\partial}{\partial\lambda}(\lambda V_\nu) + \frac{V_\mu}{\lambda}\frac{\sqrt{(\mu^2-b^2)(c^2-\mu^2)}}{\mu^2-\nu^2}\frac{\partial}{\partial\mu}\left(\sqrt{\mu^2-\nu^2}\,V_\nu\right) +$$

$$\frac{V_\nu}{\lambda}\sqrt{\frac{(b^2-\nu^2)(c^2-\nu^2)}{\mu^2-\nu^2}}\frac{\partial V_\nu}{\partial\nu} +$$

$$\frac{V_\mu^{\,2}\nu}{\lambda(\mu^2-\nu^2)}\sqrt{\frac{(b^2-\nu^2)(c^2-\nu^2)}{\mu^2-\nu^2}}\,, \qquad (253)$$

$$\frac{g^{jk}}{\sqrt{g_{11}}}\,\Upsilon_{1j,k} = -\frac{\partial p}{\partial\lambda} + \frac{1}{\lambda^2}\frac{\partial}{\partial\lambda}\left(\lambda^2\hat{\pi}^{\lambda\lambda}\right) +$$

$$\frac{\sqrt{(\mu^2-b^2)(c^2-\mu^2)}}{\lambda(\mu^2-\nu^2)}\frac{\partial}{\partial\mu}\left(\sqrt{\mu^2-\nu^2}\,\hat{\pi}^{\lambda\mu}\right) +$$

$$\frac{\sqrt{(b^2-\nu^2)(c^2-\nu^2)}}{\lambda(\mu^2-\nu^2)}\frac{\partial}{\partial\nu}\left(\sqrt{\mu^2-\nu^2}\,\hat{\pi}^{\lambda\nu}\right) -$$

$$\frac{1}{\lambda}\hat{\pi}^{\mu\mu} - \frac{1}{\lambda}\hat{\pi}^{\nu\nu}\,, \qquad (254)$$

$$\frac{g^{jk}}{\sqrt{g_{22}}}\,T_{2j,k} = -\frac{1}{\lambda}\sqrt{\frac{(\mu^2-b^2)(c^2-\mu^2)}{\mu^2-\nu^2}}\,\frac{\partial p}{\partial\mu} + \frac{1}{\lambda}\frac{\partial}{\partial\lambda}\left(\lambda\hat{\Pi}^{\mu\lambda}\right) +$$

$$\frac{1}{\lambda\sqrt{\mu^2-\nu^2}}\frac{\partial}{\partial\mu}\left(\sqrt{(\mu^2-b^2)(c^2-\mu^2)}\;\hat{\Pi}^{\mu\mu}\right) +$$

$$\frac{1}{\lambda}\sqrt{\frac{(b^2-\nu^2)(c^2-\nu^2)}{\mu^2-\nu^2}}\,\frac{\partial}{\partial\nu}\left(\hat{\Pi}^{\mu\nu}\right) + \frac{\mu\sqrt{(\mu^2-b^2)(c^2-\mu^2)}}{\lambda(\mu^2-\nu^2)^{3/2}}\,\hat{\Pi}^{\mu\mu} -$$

$$\frac{\mu\,(b^2+c^2-2\mu^2)}{\lambda\sqrt{(\mu^2-b^2)(c^2-\mu^2)(\mu^2-\nu^2)}}\,\hat{\Pi}^{\mu\mu} -$$

$$\frac{\mu\sqrt{(\mu^2-b^2)(c^2-\mu^2)}}{\lambda\,(\mu^2-\nu^2)^{3/2}}\,\hat{\Pi}^{\nu\nu} + \frac{2}{\lambda}\,\hat{\Pi}^{\mu\lambda} -$$

$$\frac{2\nu\sqrt{(b^2-\nu^2)(c^2-\nu^2)}}{\lambda\,(\mu^2-\nu^2)^{3/2}}\,\hat{\Pi}^{\mu\nu}, \tag{255}$$

$$\frac{g^{jk}}{\sqrt{g_{33}}}\,T_{3j,k} = -\frac{1}{\lambda}\sqrt{\frac{(b^2-\nu)(c^2-\nu)}{\mu^2-\nu^2}}\,\frac{\partial p}{\partial\nu} + \frac{1}{\lambda}\frac{\partial}{\partial\lambda}\left(\lambda\hat{\Pi}^{\nu\lambda}\right) +$$

$$\frac{1}{\lambda}\sqrt{\frac{(\mu^2-b^2)(c^2-\mu^2)}{\mu^2-\nu^2}}\,\frac{\partial}{\partial\mu}\left(\hat{\Pi}^{\nu\mu}\right) +$$

$$\frac{1}{\lambda\sqrt{\mu^2-\nu^2}}\frac{\partial}{\partial\nu}\left(\sqrt{(b^2-\nu^2)(c^2-\nu^2)}\;\hat{\Pi}^{\nu\nu}\right) +$$

$$\frac{\nu\sqrt{(b^2-\nu^2)(c^2-\nu^2)}}{\lambda(\mu^2-\nu^2)^{3/2}}\,\hat{\Pi}^{\mu\mu} - \frac{\nu\sqrt{(b^2-\nu^2)(c^2-\nu^2)}}{\lambda\,(\mu^2-\nu^2)^{3/2}}\,\hat{\Pi}^{\nu\nu} +$$

$$\frac{\nu(b^2+c^2-2\nu^2)}{\lambda\sqrt{(b^2-\nu^2)(c^2-\nu^2)(\mu^2-\nu^2)}}\,\hat{\Pi}^{\nu\nu} + \frac{2}{\lambda}\,\hat{\Pi}^{\nu\lambda} +$$

$$\frac{2\mu\sqrt{(\mu^2-b^2)(c^2-\mu^2)}}{\lambda(\mu^2-\nu^2)^{3/2}}\,\hat{\Pi}^{\nu\mu}\,. \tag{256}$$

IV. Conclusions

A report of this nature has no conclusions in the usual sense of the word. The methods used in arriving at the final results were not new. It was known at the outset that they would produce valid results if they were applied correctly. This project was undertaken because the equations of fluid mechanics are not readily available except in terms of three or four of the more commonly used coordinate systems. The tabulated equations in Section III are, therefore, really the "conclusions" of this thesis.

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Appendix A

Description of Coordinate Systems

The coordinate transformations presented here were taken, with minor changes, from the work of Chester H. Page (Ref 3: 98-107).

(1) Cartesian coordinates

$$x = x \qquad -\infty \le x \le \infty$$

$$y = y \qquad -\infty \le y \le \infty$$

$$z = z \qquad -\infty \le z \le \infty$$

$$g_{11} = g_{22} = g_{33} = 1$$

and the surfaces x, y, or z = constant are planes.

(2) Cylindrical polar coordinates

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$

with the variable ranges

$$-\infty \le z \le \infty$$

$$0 \le r \le \infty$$

$$0 \le \theta \le 2\pi.$$

The surfaces are:

z = const ~ planes

r = const ~ cylinders

θ = const ~ planes through the z axis.

We have:

$$g_{11} = g_{33} = 1, \quad g_{22} = r^2.$$

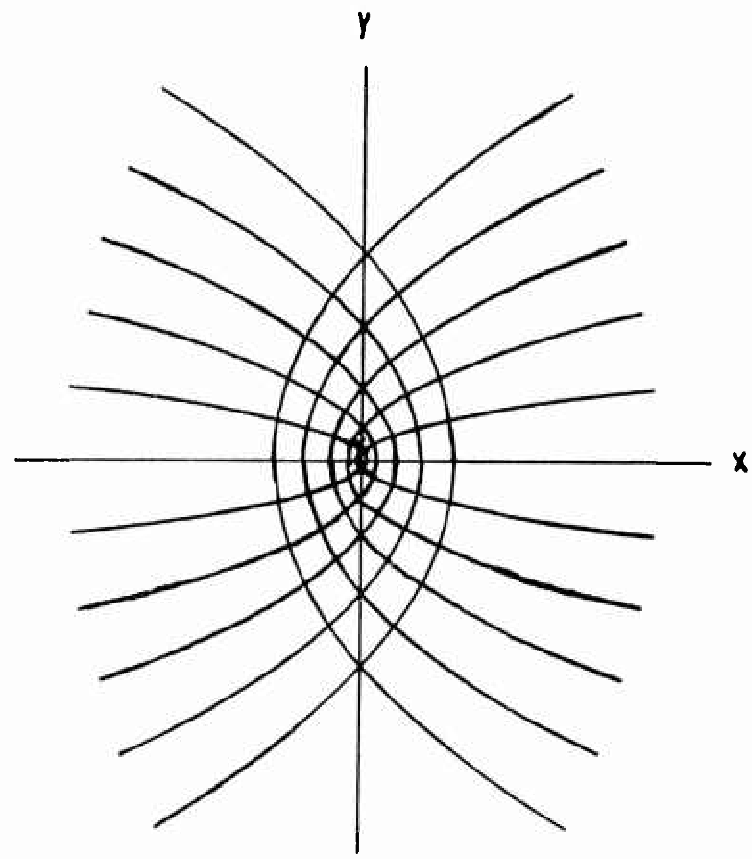

Fig. 1

Confocal Parabolas
with a Common Axis

(3) Spherical polar coordinates

$$x = r \sin\theta \cos\varphi$$

$$y = r \sin\theta \sin\varphi$$

$$z = r \cos\theta$$

$0 \le r \le \infty$, r = const ~ spheres

$0 \le \varphi \le 2\pi$, φ = const ~ azimuthal planes

$0 \le \theta \le \pi$, θ = const ~ circular cones

$g_{11} = 1$, $g_{22} = r^2$, $g_{33} = r^2 \sin^2\theta$

(4) Parabolic coordinates

Two sets of coordinate surfaces are generated by rotating the parabolas of Fig. 1 about the x axis which is then renamed the z axis. The third set of coordinate surfaces are azimuthal planes through the newly relabeled z axis.

The transformations are:

$$x = \sqrt{\lambda\mu}\cos\varphi$$

$$y = \sqrt{\lambda\mu}\sin\varphi$$

$$z = \frac{\lambda-\mu}{2}$$

$0 \le \lambda \le \infty$, λ = const ~ paraboloids:

$$x^2 + y + 2\lambda z = \lambda^2$$

$0 \le \mu \le \infty$, μ = const ~ paraboloids:

$$x^2 + y - 2\mu z = \mu^2$$

$0 \le \varphi \le 2\pi$, φ = const ~ azimuthal planes

$$g_{11} = \frac{\lambda+\mu}{4\lambda}, \quad g_{22} = \frac{\lambda+\mu}{4\mu}, \quad g_{33} = \lambda\mu$$

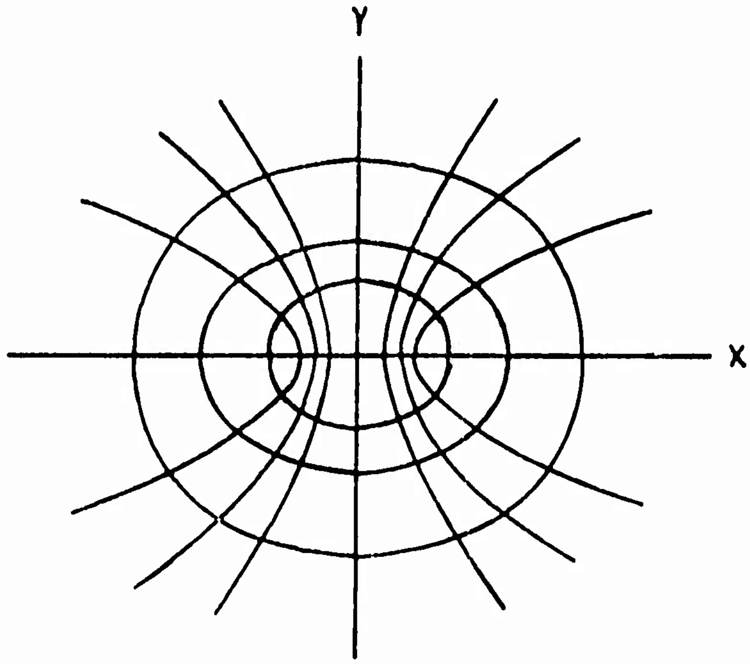

Fig. 2

Confocal Ellipses and Hyperbolas

(5) Prolate spheroidal coordinates

This coordinate system is constructed by rotating the curves of Fig. 2 about the x axis which is then renamed the z axis. The coordinate surfaces which are generated are prolate spheroids and hyperboloids of two sheets. The third set of coordinate surfaces are azimuthal planes through the newly renamed z axis.

If we let $\lambda \equiv \cosh \alpha$, $\mu = \cos \beta$, we have

$$x = a\sqrt{\lambda^2 - 1}\sqrt{1-\mu^2}\ \cos\phi$$
$$y = a\sqrt{\lambda^2 - 1}\sqrt{1-\mu^2}\ \sin\phi$$
$$z = a\lambda\mu$$

$1 \leq \lambda \leq \infty$, λ = const $\sim$ prolate spheroids:

$$\frac{x^2+y^2}{\lambda^2-1} + \frac{z^2}{\lambda^2} = a^2$$

$-1 \leq \mu \leq 1$, μ = const $\sim$ hyperloids of two sheets:

$$\frac{z^2}{\mu^2} - \frac{x^2+y^2}{1-\mu^2} = a^2$$

$0 \leq \varphi \leq 2\pi$, φ = const $\sim$ azimuthal planes

$$g_{11} = a^2\frac{\lambda^2-\mu^2}{\lambda^2-1}, \quad g_{22} = a^2\frac{\lambda^2-\mu^2}{1-\mu^2}, \quad g_{33} = a^2(\lambda^2-1)(1-\mu^2)$$

(6) Spheroidal coordinates (oblate spheroids)

This coordinate system is constructed by rotating the curves of Fig. 2 about the y axis which is then renamed the z axis. The generated coordinate surfaces are oblate spheroids and hyperboloids of one sheet. The third set of coordinate surfaces are azimuthal planes through the new z axis.

$$x = a\lambda\mu\cos\varphi \quad \text{or} \quad r = a\cosh\alpha\cos\beta$$
$$y = a\lambda\mu\sin\varphi \qquad\qquad z = a\sinh\alpha\sin\beta$$
$$z = a\sqrt{(\lambda^2-1)(1-\mu^2)}$$

λ = const $\sim$ oblate spheroids:

$$\frac{x^2+y^2}{\lambda^2} + \frac{z^2}{\lambda^2-1} = a^2$$

$\mu =$ const $\sim$ hyperboloids of one sheet:

$$\frac{x^2+y^2}{\mu^2} - \frac{z^2}{1-\mu^2} = a^2$$

$\varphi =$ const $\sim$ azimuthal planes

with the variable ranges

$$-\infty \le \alpha \le \infty \quad , \quad 1 \le \lambda \le \infty \, ,$$
$$0 \le \beta \le \frac{\pi}{2} \quad , \quad 0 \le \mu \le 1 \, ,$$
$$0 \le \varphi \le 2\pi \, .$$

We have

$$g_{11} = a^2 \frac{\lambda^2-\mu^2}{\lambda^2-1} \, , \quad g_{22} = a^2 \frac{\lambda^2-\mu^2}{1-\mu^2} \, , \quad g_{33} = a^2\lambda^2\mu^2$$

(7) Parabolic cylinder coordinates

The traces of the coordinate surfaces of this coordinate system on the xy plane are shown in Fig. 1.

$$x = \frac{\lambda-\mu}{2}$$

$$y = \sqrt{\lambda\mu}$$

$$z = z$$

$0 \le \lambda \le \infty$, $\lambda =$ const $\sim$ parabolic cylinders, $y^2 + 2\lambda x = \lambda^2$

$0 \le \mu \le \infty$, $\mu =$ const $\sim$ parabolic cylinders, $y^2 - 2\mu x = \mu^2$

$-\infty \le z \le \infty$, $z =$ const $\sim$ planes

$$g_{11} = \frac{\lambda+\mu}{4\lambda} \, , \quad g_{22} = \frac{\lambda+\mu}{4\mu} \, , \quad g_{33} = 1$$

(8) Elliptic cylinder coordinates

The traces of the coordinate surfaces of this coordinate system on the xy plane are shown in Fig. 2.

$$x = a\sqrt{\lambda^2-1}\sqrt{1-\mu^2} \qquad \text{or} \qquad x = a \sinh\alpha \sin\beta$$

$$y = a\lambda\mu \qquad\qquad y = a\cosh\alpha\cos\beta$$

$$z = z \qquad\qquad z = z$$

λ = const $\sim$ elliptic cylinders, $\dfrac{x^2}{\lambda^2-1} + \dfrac{y^2}{\lambda^2} = a^2$

μ = const $\sim$ hyperbolic cylinders, $\dfrac{y^2}{\mu^2} - \dfrac{x^2}{1-\mu^2} = a^2$

z = const $\sim$ planes

with the variable ranges

$$-\infty \le \alpha \le +\infty, \qquad 1 \le \lambda \le \infty,$$

$$0 \le \beta \le \pi, \qquad -1 \le \mu \le 1,$$

$$-\infty \le z \le \infty.$$

We have

$$g_{11} = a^2\frac{\lambda^2-\mu^2}{\lambda^2-1}\ ; \quad g_{22}\ a^2\frac{\lambda^2-\mu^2}{1-\mu^2}, \quad g_{33} = 1$$

(9) Ellipsoidal coordinates

$$x^2 = \frac{(a^2-\lambda)(a^2-\mu)(a^2-\nu)}{(a^2-b^2)(a^2-c^2)} \qquad \text{with} \quad -\infty < \lambda \le c^2 < \mu < b^2 \le \nu < a^2$$

$$y^2 = \frac{(b^2-\lambda)(b^2-\mu)(b^2-\nu)}{(b^2-c^2)(a^2-b^2)} \qquad +$$

$$z^2 = \frac{(c^2-\lambda)(c^2-\mu)(c^2-\nu)}{(a^2-c^2)(b^2-c^2)}$$

The surfaces are:

λ = const $\sim$ ellipsoid:

$$\frac{x^2}{a^2-\lambda} + \frac{y^2}{b^2-\lambda} + \frac{z^2}{c^2-\lambda} = 1, \quad a > b > c, \quad -\infty \leq \lambda \leq c^2,$$

μ = const $\sim$ hyperboloid of one sheet:

$$\frac{x^2}{a^2-\mu} + \frac{y^2}{b^2-\mu} + \frac{z^2}{c^2-\mu} = 1, \quad c^2 \leq \mu \leq b^2,$$

ν = const $\sim$ hyperboloid of two sheets:

$$\frac{x^2}{a^2-\nu} + \frac{y^2}{b^2-\nu} + \frac{z^2}{c^2-\nu} = 1, \quad b^2 \leq \nu \leq a^2 .$$

Also,

$$g_{11} = \frac{1}{4} \frac{(\mu-\lambda)(\nu-\lambda)}{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)},$$

$$g_{22} = \frac{1}{4} \frac{(\lambda-\mu)(\nu-\mu)}{(a^2-\mu)(b^2-\mu)(c^2-\mu)},$$

and

$$g_{33} = \frac{1}{4} \frac{(\lambda-\nu)(\mu-\nu)}{(a^2-\nu)(b^2-\nu)(c^2-\nu)} .$$

(10) <u>Confocal parabolic coordinates</u>

The transformations are:

$$x = \frac{1}{2}(\lambda + \mu + \nu - a - b)$$

$$y^2 = \frac{(a-\lambda)(a-\mu)(a-\nu)}{b-a}$$

$$z^2 = \frac{(b-\lambda)(b-\mu)(b-\nu)}{a-b} .$$

The surfaces λ, μ, and ν constant are the paraboloids

$$\frac{2x}{\lambda} + \frac{y^2}{\lambda(\lambda-a)} + \frac{z^2}{\lambda(\lambda-b)} = 1$$

$$\frac{2x}{\mu} + \frac{y^2}{\mu(\mu-a)} + \frac{z^2}{\mu(\mu-b)} = 1$$

$$\frac{2x}{\nu} + \frac{y^2}{\nu(\nu-a)} + \frac{z^2}{\nu(\nu-b)} = 1$$

with

$$-\infty < \nu < a < \mu < b < \lambda < \infty .$$

Also,

$$g_{11} = \frac{1}{4}\,\frac{(\lambda-\mu)(\lambda-\nu)}{(a-\lambda)(b-\lambda)} ,$$

$$g_{22} = \frac{1}{4}\,\frac{(\mu-\lambda)(\mu-\nu)}{(a-\mu)(b-\mu)} ,$$

and

$$g_{33} = \frac{1}{4}\,\frac{(\nu-\lambda)(\nu-\mu)}{(a-\nu)(b-\nu)} .$$

(11) Conical coordinates

The surfaces λ, μ, and ν constant are the spheres $x^2 + y^2 + z^2 = \lambda^2$ and the two cones

$$\frac{x^2}{\mu^2} + \frac{y^2}{\mu^2 - b^2} + \frac{z^2}{\mu^2 - c^2} = 0 \qquad \text{and}$$

$$\frac{x^2}{\nu^2} + \frac{y^2}{\nu^2 - b^2} + \frac{z^2}{\nu^2 - c^2} = 0 ,$$

where $\quad 0 \le \nu \le b \le \mu \le c \quad , \quad 0 \le \lambda \le \infty .$

The cones, μ = constant, intersect the planes, z = constant, in ellipses. The cones, ν = constant, intersect these z planes in hyperbolas, but intersect the planes, x = constant, in ellipses. Hence, we can visualize the cones as being elliptical cones centered about the z and x axes.

The transformations are

$$x^2 = \frac{\lambda^2 \mu^2 \nu^2}{b^2 c^2} ,$$

$$y^2 = \frac{\lambda^2(\mu^2 - b^2)(\nu^2 - b^2)}{b^2(b^2 - c^2)} \quad , \quad \text{and}$$

$$z^2 = \frac{\lambda^2(\mu^2 - c^2)(\nu - c^2)}{c^2(c^2 - b^2)} \; .$$

Also,

$$g_{11} = 1 \quad ,$$

$$g_{22} = \frac{\lambda^2(\mu^2 - \nu^2)}{(\mu^2 - b^2)(c^2 - \mu^2)} \quad ,$$

and

$$g_{33} = \frac{\lambda^2(\mu^2 - \nu^2)}{(b^2 - \nu^2)(c^2 - \nu^2)} \; .$$

Appendix B

Derivation of Basic Equations of Fluid Mechanics

The derivations shown here can be found, in slightly varying form, in the works of Sokolnikoff (Ref 4: 290-324) and McConnell (Ref 1: 270-282), among others. These specific forms were, however, first shown to the author by Lt. Ray M. Bowen of the Mechanical Engineering Department of the Institute of Technology.

Equation of Continuity

The equation of continuity can be expressed in integral form as follows:

$$\frac{d}{dt}\int_{V(t)} \rho(x^i,t)\,d\tau = 0 \; . \tag{B.1}$$

Application of Leibnitz's Rule yields

$$\int_{V(t)} \frac{\partial \rho}{\partial t}\,d\tau + \oint_{S(V)} \rho v^i n_i\,ds = 0 . \tag{B.2}$$

By the Divergence Theorem

$$\int_{V(t)} \frac{\partial \rho}{\partial t}\,d\tau + \int_{V(t)} (\rho v^i)_{,i}\,d\tau = 0 . \tag{B.3}$$

Then, since volume is arbitrary,

$$\frac{\partial \rho}{\partial t} + (\rho v^i)_{,i} = 0 , \tag{B.4}$$

but

$$(\rho v^i)_{,i} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\sqrt{g}\,\rho\, v^i\right) . \qquad \text{(B.5)}$$

(Ref 1: 155). The continuity equation then becomes

$$\frac{\partial \rho}{\partial t} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\sqrt{g}\,\rho\, v^i\right) = 0 . \qquad \text{(B.6)}$$

The physical components of the contravariant quantity v^i are represented by $\sqrt{g_{(i)(i)}}\,v^i$. If we denote the physical components by the symbol V_i, then $v^i = \frac{V_i}{\sqrt{g_{(i)(i)}}}$. We can then write the continuity equation in its final form:

$$\frac{\partial \rho}{\partial t} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\frac{\sqrt{g}}{\sqrt{g_{(i)(i)}}}\,\rho\, V_i\right) = 0 . \qquad \text{(B.7)}$$

Energy Equation

The energy equation can be written in integral form as follows:

$$\frac{d}{dt}\int_{V(t)} \rho\left(u + \tfrac{1}{2} v_i v^i\right) d\tau = \oint_{S(V)} v_i \tau^{ij} n_j \, dS + \int_{V(t)} v_i \rho F^i d\tau + \oint_{S(V)} q^i(-n_i)\, dS . \qquad \text{(B.8)}$$

Application of Leibnitz's Rule yields

$$\int_{V(t)} \frac{\partial}{\partial t}\left[\rho\left(u+\tfrac{1}{2}v_i v^i\right)\right] d\tau + \oint_{S(v)} \left[\rho\left(u+\tfrac{1}{2}v_i v^i\right)v^j n_j\right] ds =$$

$$\oint_{S(v)} v_i \tau^{ij} n_j \, ds + \int_{V(t)} v_i \rho F^i d\tau - \oint_{S(v)} q^i n_i \, ds , \qquad \text{(B.9)}$$

and, by the Divergence Theorem,

$$\int_{V(t)} \left\{ \frac{\partial}{\partial t}\left[\rho\left(u+\tfrac{1}{2}v_i v^i\right)\right] + \left[\rho\left(u+\tfrac{1}{2}v_i v^i\right)v^j\right]_{,j} \right\} d\tau =$$

$$\int_{V(t)} \left[(v_i \tau^{ij})_{,j} + \rho v_i F^i - q^i{}_{,i} \right] d\tau \qquad \text{(B.10)}$$

Then, since the volume is arbitrary,

$$\frac{\partial}{\partial t}\left[\rho\left(u+\tfrac{1}{2}v_i v^i\right)\right] + \left[\rho\left(u+\tfrac{1}{2}v_i v^i\right)v^j\right]_{,j} =$$

$$(v_i \tau^{ij})_{,j} + \rho v_i F^i - q^i{}_{,i} \, . \qquad \text{(B.11)}$$

By expanding and subtracting out the continuity equation we obtain

$$\rho\left[\frac{\partial}{\partial t}\left(u+\tfrac{1}{2}v_i v^i\right) + v^j\left(u+\tfrac{1}{2}v_i v^i\right)_{,j}\right] =$$

$$(v_i \tau^{ij})_{,j} + \rho v_i F^i - q^i{}_{,i} \, . \qquad \text{(B.12)}$$

But

$$u = h - \frac{p}{\rho} \tag{B.13}$$

Substitution of (B.13) into (B.12) yields

$$\rho\left[\frac{\partial}{\partial t}\left(h + \tfrac{1}{2} v_i v^i\right) + v^j\left(h + \tfrac{1}{2} v^i v_i\right)_{,j}\right] -$$

$$\rho\left[\frac{\partial}{\partial t}\left(p/\rho\right) + v^j\left(p/\rho\right)_{,j}\right] = -\left(p v^j\right)_{,j} +$$

$$\left(v_i \pi^{ij}\right)_{,j} + \rho v_i F^i - g^i_{,i} \,, \tag{B.14}$$

where we have also made use of the relationship

$$\tau^{ij} = -p g^{ij} + \pi^{ij} \,. \tag{B.15}$$

After simplifying and rearranging, the energy equation becomes

$$\rho\left[\frac{\partial}{\partial t}\left(h + \tfrac{1}{2} v_i v^i\right) + v^j\left(h + \tfrac{1}{2} v_i v^i\right)_{,j}\right] =$$

$$\frac{\partial p}{\partial t} + \left(v_i \pi^{ij}\right)_{,j} + \rho v_i F^i - g^i_{,i} \,, \tag{B.16}$$

or, in terms of physical components

$$\rho\left[\frac{\partial}{\partial t}\left(h+\frac{1}{2}V_iV_i\right)+\frac{V_j}{\sqrt{g_{(j)(j)}}}\frac{\partial}{\partial x^j}\left(h+\frac{1}{2}V_iV_i\right)\right]=\frac{\partial p}{\partial t}+$$

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(\frac{\sqrt{g}}{\sqrt{g_{(j)(j)}}}V_i\hat{\pi}^{ij}\right)+\rho V_i\hat{F}_i-\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\frac{\sqrt{g}}{\sqrt{g_{(i)(i)}}}\hat{q}_i\right) \quad \text{(B.17)}$$

Equation of Motion

The equation of motion can be written in integral form as follows:

$$\frac{d}{dt}\int_{V(t)}\rho v^i d\tau=\oint_{S(V)}\tau^{ij}n_j dS+\int_{V(t)}\rho F^i d\tau, \qquad \text{(B.18)}$$

where

$$\tau^{ij}=-pg^{ij}+\pi^{ij}. \qquad \text{(B.15)}$$

By Leibnitz's Rule

$$\int_{V(t)}\frac{\partial}{\partial t}(\rho v^i)\,d\tau+\oint_{S(V)}\rho v^i v^j n_j\,dS=$$

$$\oint_{S(V)}\tau^{ij}n_j\,dS+\int_{V(t)}\rho F^i\,d\tau, \qquad \text{(B.19)}$$

and, by the Divergence Theorem,

$$\int_{V(t)} \left[\frac{\partial}{\partial t} (\rho v^i) + (\rho v^i v^j)_{,j} \right] d\tau = \int_{V(t)} (\tau^{ij}_{\ ,j} + \rho F^i) d\tau. \tag{B.20}$$

Then, since the volume is arbitrary,

$$\frac{\partial}{\partial t} (\rho v^i) + (\rho v^i v^j)_{,j} = \tau^{ij}_{\ ,j} + \rho F^i. \tag{B.21}$$

Now, after expanding and subtracting out the continuity equation, we have

$$\rho \left(\frac{\partial v^i}{\partial t} + v^i_{\ ,j} v^j \right) = \tau^{ij}_{\ ,j} + \rho F^i, \tag{B.22}$$

or

$$\rho f^i - \rho F^i = \tau^{ij}_{\ ,j} , \tag{B.23}$$

where

$$f^i \equiv \frac{\partial v^i}{\partial t} + v^i_{\ ,j} v^j = \frac{\delta v^i}{\delta t} . \tag{B.24}$$

Equation (B.23) can be expressed in covariant form as

$$g^{jk} \tau_{ij,k} = \rho (f_i - F_i) \tag{B.25}$$

Appendix C

The Three Basic Equations

Continuity Equation

$$\frac{\partial \rho}{\partial t} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\frac{\sqrt{g}}{\sqrt{g_{(i)(i)}}} \rho V_i \right) = 0 \tag{3}$$

Energy Equation

$$\rho \left[\frac{\partial}{\partial t}\left(h + \frac{1}{2} V_i V_i \right) + \frac{V_j}{\sqrt{g_{(j)(j)}}} \frac{\partial}{\partial x^j}\left(h + \frac{1}{2} V_i V_i \right) \right] = \frac{\partial p}{\partial t} +$$

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j}\left(\frac{\sqrt{g}}{\sqrt{g_{(j)(j)}}} V_i \hat{\pi}^{ij} \right) + \rho V_i \hat{F}_i - \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\frac{\sqrt{g}}{\sqrt{g_{(i)(i)}}} \hat{q}_i \right) \tag{5}$$

Equation of Motion

$$\frac{g^{jk}}{\sqrt{g_{(i)(i)}}} \tau_{(i)j,k} = \rho \left(\hat{f}_i - \hat{F}_i \right) \tag{12}$$

Vita

Bradley Sutter was born [redacted] [redacted] the son of [redacted] [redacted] He graduated from [redacted] in [redacted] in 1948, and spent one year at the University of Minnesota before entering the United States Naval Academy in June, 1949. He was graduated from the Academy with the degree of Bachelor of Science in June, 1953. Since completing pilot training in August, 1954, he has served as a flight instructor in the Air Training Command and as a pilot in the Strategic Air Command. He entered the Institute of Technology in February, 1961.

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